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CONVEX FUNCTIONS AND ORLICZ SPACES

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CONVEX FUNCTIONS AND ORLICZ SPACES

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DEDICATION

This translation is dedicated to Prof. Dr. M. A. Krasnosel'skiĭ and to Prof. Dr. Ya. B. Rutickiĭ.

L. F. B.

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ANNOTATION

In the present book are discussed the theory of extensive classes of convex functions which play an important role in many branches of mathematics. The theory of Orlicz spaces (i. e. normed spaces of which the L^p spaces are a special case) is developed in detail and applications are pointed out.

The book is intended for mathematicians (students in upper level courses, aspirants for the doctoral degree and scientific workers), who deal with functional analysis and its applications and also with various problems in the theory of functions.

FOREWORD

The present monograph consists of four chapters.

In the first of these chapters we study various classes of convex functions. The fundamental content of this chapter has been published, up to this time, only in various journal articles. It appears to the authors that the material in the first chapter is of interest independently of the remainder of the book inasmuch as convex functions are applied extensively in the most diversified branches of mathematics.

In the second chapter, we discuss the general theory of Orlicz spaces–spaces which are the direct extensive generalization of L^p spaces.

Here, we consider the usual problems of functional analysis with regard to applications to Orlicz spaces: completeness, separability conditions, the existence of a basis, equivalent normizations, compactness conditions, properties of linear functions, and so on. It is made clear that Orlicz spaces are, in many cases, similar to L^p spaces.

In the third chapter, we study operators and functionals defined on Orlicz spaces.

The authors found it necessary to apply Orlicz spaces in the consideration of nonlinear integral equations of the form

$$\lambda\varphi(x) = \int_0^1 k(x, y) f[\varphi(y)]dy,$$

where $f(u)$ is a function which increases more rapidly than an arbitrary power function.

The operator defined by the right member of this integral equation does not operate in any of the L^p spaces. Therefore, the investigation of the indicated integral equations by the methods of nonlinear functional analysis turns out to be difficult. The results of the second and third chapters enable us to investigate wide classes of nonlinear equations.

The fourth, and last, chapter of the book is devoted to the investigation of some nonlinear problems.

The authors take this opportunity to express their gratitude to G. E. Šilov whose many constructive criticisms were used in preparing the present book.

CHAPTER I

SPECIAL CLASSES OF CONVEX FUNCTIONS

§ 1. *N*-functions

1. *Convex functions.* A real-valued function $M(u)$ of the real variable u is said to be *convex* if the inequality

$$M\left(\frac{u_1 + u_2}{2}\right) \leqslant \tfrac{1}{2}[M(u_1) + M(u_2)] \tag{1.1}$$

is satisfied for all values of u_1 and u_2.

We shall be interested only in continuous convex functions. Condition (1.1) means that the midpoint of the chord connecting

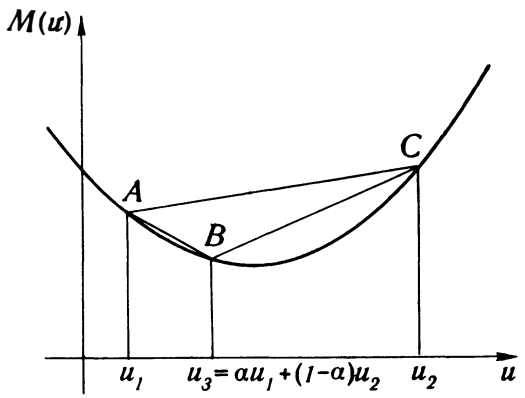

Fig. 1.

two points on the graph of the function $M(u)$ lies above the corresponding point of the graph. It is geometrically clear (see Fig. 1) that every chord lies above the graph of the function, i.e. that the inequality

$$M[\alpha u_1 + (1 - \alpha)u_2] \leqslant \alpha M(u_1) + (1 - \alpha)M(u_2) \tag{1.2}$$

is satisfied for all α $(0 \leqslant \alpha \leqslant 1)$. This inequality is called *Jensen's inequality*. Jensen's inequality can also be proved analytically.

In fact, let us assume that inequality (1.2) is not satisfied for all α in [0, 1]. Then the maximum value M_0 of the continuous function

$$f(\alpha) = M[\alpha u_1 + (1 - \alpha)u_2] - \alpha M(u_1) - (1 - \alpha)M(u_2)$$

on [0, 1] will be positive: $M_0 > 0$. We denote by α_0 the least value of the argument for which $f(\alpha)$ assumes the value M_0. Let $\delta > 0$ be a number such that the segment $[\alpha_0 - \delta, \alpha_0 + \delta]$ is contained in [0, 1]. Applying inequality (1.1) to the points

$$u_1^* = (\alpha_0 - \delta)u_1 + (1 - \alpha_0 + \delta)u_2,$$

$$u_2^* = (\alpha_0 + \delta)u_1 + (1 - \alpha_0 - \delta)u_2$$

and going over to the function $f(\alpha)$, we obtain that

$$f(\alpha_0) \leqslant \frac{f(\alpha_0 - \delta) + f(\alpha_0 + \delta)}{2} < M_0.$$

We have arrived at a contradiction which proves inequality (1.2).

If $u_1 \neq u_2$, then equality is attained in (1.2) either for $\alpha = 0$ and $\alpha = 1$ only or for all $\alpha \in [0, 1]$. In fact, suppose equality is attained in (1.2) for some $\alpha_0 \in (0, 1)$. This means that $f(\alpha_0) = 0$. We shall show that in this case $f(\alpha) = 0$ for all $\alpha \in [0, 1]$. It is easily verified that the continuous function $f(\alpha)$ is convex. Therefore it also satisfies Jensen's inequality. We assume that for some $\alpha_1 \in (0, 1)$ we have $f(\alpha_1) < 0$ (by what we have already proved, $f(\alpha)$ cannot be positive). We assume, for definiteness, that $\alpha_1 < \alpha_0$. Since

$$\alpha_0 = \frac{1 - \alpha_0}{1 - \alpha_1}\alpha_1 + \frac{\alpha_0 - \alpha_1}{1 - \alpha_1},$$

we have by Jensen's inequality that

$$f(\alpha_0) \leqslant \frac{1 - \alpha_0}{1 - \alpha_1} f(\alpha_1) - \frac{\alpha_0 - \alpha_1}{1 - \alpha_1} f(1) = \frac{1 - \alpha_0}{1 - \alpha_1} f(\alpha_1) < 0,$$

which contradicts the assumption that $f(\alpha_0) = 0$.

Inequality (1.1) admits still another generalization:

$$M\left(\frac{u_1 + u_2 + \cdots + u_n}{n}\right) \leqslant \frac{1}{n}[M(u_1) + M(u_2) + \cdots + M(u_n)] \qquad (1.3)$$

for arbitrary $u_1, u_2, \cdots, u_n$. By successive application of (1.1),

inequality (1.3) is proved for all n of the form 2^k. The case of arbitrary n is more complicated. Let m be a number such that $n + m = 2^k$. Then

$$M\left(\frac{u_1 + u_2 + \cdots + u_n + mu^*}{n+m}\right) \leqslant$$

$$\leqslant \frac{1}{n+m}\,[M(u_1) + M(u_2) + \cdots + M(u_n) + mM(u^*)].$$

Setting $u^* = \dfrac{u_1 + u_2 + \cdots + u_n}{n}$, we obtain (1.3).

We shall assume that $u_1 \leqslant u_3 \leqslant u_2$. Then

$$u_3 = \frac{u_2 - u_3}{u_2 - u_1}\,u_1 + \frac{u_3 - u_1}{u_2 - u_1}\,u_2,$$

and, in virtue of inequality (1.2), we have that

$$M(u_3) \leqslant \frac{u_2 - u_3}{u_2 - u_1}\,M(u_1) + \frac{u_3 - u_1}{u_2 - u_1}\,M(u_2),$$

from which it follows that

$$\frac{M(u_3) - M(u_1)}{u_3 - u_1} \leqslant \frac{M(u_2) - M(u_1)}{u_2 - u_1} \leqslant \frac{M(u_2) - M(u_3)}{u_2 - u_3}. \tag{1.4}$$

The inequalities obtained mean that the slope of the chord AB is less than the slope of the chord AC which, in turn, is less than the slope of the chord BC.

2. *Integral representation of a convex function.*

LEMMA 1.1. *A continuous convex function $M(u)$ has, at every point, a right derivative $p_+(u)$ and a left derivative $p_-(u)$ such that*

$$p_-(u) \leqslant p_+(u). \tag{1.5}$$

PROOF. In virtue of (1.4), we have that

$$\frac{M(u) - M(u - h_2)}{h_2} \leqslant \frac{M(u) - M(u - h_1)}{h_1} \leqslant$$

$$\leqslant \frac{M(u + h_1) - M(u)}{h_1} \leqslant \frac{M(u + h_2) - M(u)}{h_2} \tag{1.6}$$

for $0 < h_1 < h_2$.

It follows from these inequalities that the ratio

$$\frac{M(u) - M(u-h)}{h}$$

does not decrease as $h \to +0$ and, consequently, it has a limit $p_-(u)$. Analogously, the ratio

$$\frac{M(u+h) - M(u)}{h}$$

does not increase as $h \to +0$ and has a limit $p_+(u)$. Inequality (1.5) also follows from (1.6).

LEMMA 1.2. *The right derivative $p_+(u)$ of a continuous convex function $M(u)$ is a non-decreasing right-continuous function.*

PROOF. Let $u_1 < u_2$. Then, for sufficiently small positive h, we have that $u_1 + h < u_2 - h$ and, in virtue of (1.4),

$$\frac{M(u_1+h) - M(u_1)}{h} \leqslant \frac{M(u_2) - M(u_2-h)}{h}.$$

Passing to the limit, we obtain

$$p_+(u_1) \leqslant p_-(u_2). \tag{1.7}$$

From this inequality and (1.5) it follows that

$$p_+(u_1) \leqslant p_+(u_2). \tag{1.8}$$

The monotonicity of the function $p_+(u)$ is thus proved.

As was shown in the course of the proof of Lemma 1.1, for all $h > 0$, we have

$$p_+(u) \leqslant \frac{M(u+h) - M(u)}{h}.$$

Keeping h fixed and passing to the limit as $u \to u_0 + 0$, we obtain

$$\lim_{u \to u_0 + 0} p_+(u) \leqslant \frac{M(u_0+h) - M(u_0)}{h} \tag{1.9}$$

in virtue of the continuity of the function $M(u)$. The limit in the left member of this inequality exists in virtue of the monotonicity

of the function $p_+(u)$. Passing to the limit as $h \to +0$ in (1.9), we obtain

$$\lim_{u \to u_0 + 0} p_+(u) \leqslant p_+(u_0).$$

On the other hand, $p_+(u) \geqslant p_+(u_0)$ for $u \geqslant u_0$ in virtue of which

$$\lim_{u \to u_0 + 0} p_+(u) \geqslant p_+(u_0).$$

Thus,

$$\lim_{u \to u_0 + 0} p_+(u) = p_+(u_0).$$

This equality does indeed mean that the function $p_+(u)$ is right-continuous. ★

REMARK. It can be proved analogously that the left derivative $p_-(u)$ is a non-decreasing left-continuous function.

LEMMA 1.3. *A convex function $M(u)$ is absolutely continuous and satisfies the Lipschitz condition in every finite interval.*

PROOF. We consider any interval $[a, b]$. Let $a < u_1 < u_2 < b$. In virtue of (1.4), we have that

$$\frac{M(u_1) - M(a)}{u_1 - a} \leqslant \frac{M(u_2) - M(u_1)}{u_2 - u_1} \leqslant \frac{M(b) - M(u_2)}{b - u_2}.$$

It follows from these inequalities that

$$p_+(a) \leqslant \frac{M(u_2) - M(u_1)}{u_2 - u_1} \leqslant p_-(b),$$

i.e. that the quantity $|\{M(u_2) - M(u_1)\}/(u_2 - u_1)|$ is bounded for all u_1, u_2 in the interval $[a, b]$. ★

THEOREM 1.1. *Every convex function $M(u)$ which satisfies the condition $M(a) = 0$ can be represented in the form*

$$M(u) = \int_a^u p(t)\, dt, \tag{1.10}$$

where $p(t)$ is a non-decreasing right-continuous function.

PROOF. We note first of all that the function $M(u)$ has a derivative almost everywhere. In fact, in virtue of (1.7) and (1.5), we have that

$$p_-(u_2) \geqslant p_+(u_1) \geqslant p_-(u_1) \tag{1.11}$$

for $u_2 > u_1$. Since the function $p_-(u)$ is monotonic, it is continuous

almost everywhere. Let u_1 be a point of continuity of the function $p_-(u)$. Passing to the limit in (1.11) as $u_2 \to u_1$, we obtain that $p_-(u_1) \geqslant p_+(u_1) \geqslant p_-(u_1)$, i.e. $p_-(u_1) = p_+(u_1)$.

Similarly, we have that $M'(u) = p(u) = p_+(u)$ almost everywhere.

Since the function $M(u)$ is absolutely continuous (in virtue of Lemma 1.3), it is the indefinite integral of its derivative (see, for example, NATANSON [1]). ★

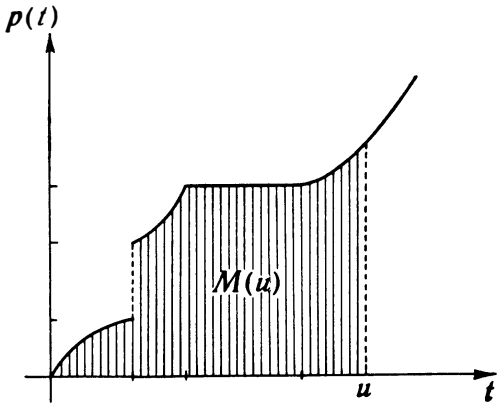

Fig. 2.

3. *Definition of an N-function.* A function $M(u)$ is called an *N-function* if it admits of the representation

$$M(u) = \int_0^{|u|} p(t)\,dt, \tag{1.12}$$

where the function $p(t)$ is right-continuous for $t \geqslant 0$, positive for $t > 0$ and non-decreasing which satisfies the conditions

$$p(0) = 0, \quad p(\infty) = \lim_{t\to\infty} p(t) = \infty. \tag{1.13}$$

Roughly speaking, the above conditions signify that the function $p(t)$ must have a graph of the form shown in Fig. 2. The value of the N-function itself is the magnitude of the area of the corresponding curvilinear trapezoid.

For example, the functions $M_1(u) = |u|^\alpha/\alpha$ $(\alpha > 1)$, $M_2(u) = e^{u^2} - 1$ are N-functions. For the first of these, $p_1(t) = M_1'(t) = t^{\alpha-1}$ and, for the second, $p_2(t) = M_2'(t) = 2te^{t^2}$.

4. *Properties of N-functions.* It follows from representation (1.12) that every N-function is even, continuous, assumes the value zero at the origin, and increases for positive values of the argument.

N-functions are convex. In fact, if $0 \leqslant u_1 < u_2$, then, in virtue of the monotonicity of the function $p(t)$, we have that

$$M\left(\frac{u_1 + u_2}{2}\right) = \int_0^{(u_1+u_2)/2} p(t)\,dt \leqslant$$

$$\leqslant \int_0^{u_1} p(t)\,dt + \tfrac{1}{2}\left[\int_{u_1}^{(u_1+u_2)/2} p(t)\,dt + \int_{(u_1+u_2)/2}^{u_2} p(t)\,dt\right] =$$

$$= \tfrac{1}{2}\left[\int_0^{u_1} p(t)\,dt + \int_0^{u_2} p(t)\,dt\right] = \tfrac{1}{2}[M(u_1) + M(u_2)].$$

In the case of arbitrary u_1, u_2, we have that

$$M\left(\frac{u_1 + u_2}{2}\right) = M\left(\frac{|u_1 + u_2|}{2}\right) \leqslant M\left(\frac{|u_1| + |u_2|}{2}\right) \leqslant$$

$$\leqslant \tfrac{1}{2}[M(u_1) + M(u_2)].$$

Setting $u_2 = 0$ in (1.2), we obtain that

$$M(\alpha u_1) \leqslant \alpha M(u_1) \quad (0 \leqslant \alpha \leqslant 1). \tag{1.14}$$

The first of conditions (1.13) signifies that

$$\lim_{u\to 0} \frac{M(u)}{u} = 0. \tag{1.15}$$

It follows from the second condition in (1.13) that

$$\lim_{u\to\infty} \frac{M(u)}{u} = \infty \tag{1.16}$$

since, for $u > 0$, we have that

$$\frac{M(u)}{u} = \frac{1}{u}\int_0^u p(t)\,dt \geqslant \frac{1}{u}\int_{u/2}^u p(t)\,dt \geqslant p\left(\frac{u}{2}\right).$$

We note that for an N-function the equality sign can hold in (1.14) only in the case when $\alpha = 0$, 1 or when $u_1 = 0$. In fact, suppose $u_1 \neq 0$ and that for some $\alpha \in (0, 1)$ the equality sign holds in (1.14). Then, in virtue of what we said in subsection 1, above, the equality sign holds in (1.14) for all $\alpha \in [0, 1]$. But then, for all $\alpha \in [0, 1]$, $M(\alpha u_1)/(\alpha u_1) = M(u_1)/u_1$. Passing to the limit as $\alpha \to 0$ in this equality, we obtain that $\lim_{\alpha \to 0} M(\alpha u_1)/(\alpha u_1) = M(u_1)/u_1$, which contradicts (1.15).

Thus,

$$M(\alpha u) < \alpha M(u) \quad (0 < \alpha < 1,\ u \neq 0). \tag{1.17}$$

It follows from this inequality that the function $M(u)/u$ is strictly increasing for positive values of u:

$$\frac{M(u')}{u'} < \frac{M(u)}{u} \qquad (0 < u' < u). \tag{1.18}$$

To prove this assertion, it suffices to set $\alpha = u'/u$ in (1.17).

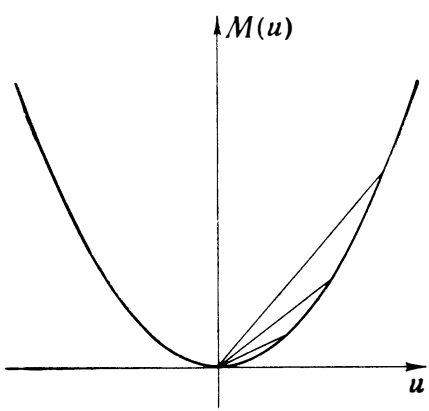

Fig. 3.

The properties established describe sufficiently well the graph of an N-function (Fig. 3). Property (1.15) signifies that the x-axis is tangent to the graph of the N-function at the origin. Properties (1.18) and (1.16) characterize the variation of the slope of the chord joining the origin with a variable point on the graph of the N-function. The graph can contain saltuses and rectilinear segments. Saltuses correspond to points of discontinuity of the function $p(t)$ and rectilinear segments correspond to its intervals of constancy.

We denote by $M^{-1}(v)$ $(0 \leqslant v < \infty)$ the inverse function to the N-function $M(u)$ considered for non-negative values of the argument. This function is convex since, in virtue of inequality (1.2), we have that $M^{-1}[\alpha v_1 + (1 - \alpha)v_2] \geqslant \alpha M^{-1}(v_1) + (1 - \alpha)M^{-1}(v_2)$ for $v_1, v_2 \geqslant 0$.

The monotonicity of the right derivative $p(u)$ of the N-function $M(u)$ implies the inequality

$$M(u) + M(v) = \int_0^{|u|} p(t)\,dt + \int_0^{|v|} p(t)\,dt \leqslant \int_0^{|u|} p(t)\,dt + \int_{|u|}^{|u|+|v|} p(t)\,dt = \int_0^{|u|+|v|} p(t)\,dt = M(|u| + |v|). \tag{1.19}$$

Suppose $a = M(u)$, $b = M(v)$ are arbitrary non-negative numbers. It then follows from (1.19) that

$$M^{-1}(a + b) \leqslant M^{-1}(a) + M^{-1}(b). \tag{1.20}$$

5. *Second definition of an N-function.* It is sometimes expedient to use the following definition: a continuous convex function $M(u)$ is called an N-function if it is even and satisfies conditions (1.15) and (1.16). We shall show that this definition is equivalent to that given above, in subsection 3. In the proof, we need only the fact that it follows from the second definition of an N-function that it is possible to represent it in the form (1.12).

In virtue of (1.15), $M(0) = 0$. Therefore, in virtue of the evenness of the function $M(u)$ and Theorem 1.1, it can be represented in the form

$$M(u) = \int_0^{|u|} p(t)\,dt,$$

where the function $p(u)$ is non-decreasing for $u > 0$ and right-continuous (i.e. the right derivative of the function $M(u)$). Since $p(u) \geqslant M(u)/u$ for $u > 0$, we have that $p(u) > 0$ for $u > 0$ and, in virtue of (1.16), $\lim_{u\to\infty} p(u) = \infty$.

On the other hand, for $u > 0$,

$$M(2u) = \int_0^{2u} p(t)\,dt > \int_u^{2u} p(t)\,dt > up(u),$$

from which it follows that $p(u) < M(2u)/u$. Therefore, in virtue of (1.15), we have that $p(0) = \lim_{u\to 0} p(u) = 0$.

6. *Composition of N-functions.* The composition $[M_1(u)]\,M_2 = M(u)$ of two N-functions $M_1(u)$ and $M_2(u)$ is also an N-function. In fact, the function $M(u)$ has (for $u > 0$) the right derivative $p(u) = p_2[M_1(u)]p_1(u)$, where $p_1(u)$, $p_2(u)$ are the right derivatives of the N-functions $M_1(u)$ and $M_2(u)$. The function $p(u)$ is right-continuous, does not decrease, and satisfies conditions (1.13) inasmuch as the functions $p_1(u)$ and $p_2(u)$ satisfy these conditions.

The converse assertion is also true: every N-function $M(u)$ is the composition $M(u) = M_2[M_1(u)]$ of two N-functions.

If the N-function $M_1(u)$ is given, then the function $M_2(u)$ is uniquely defined by the equality

$$M_2(u) = M[M_1^{-1}(|u|)], \tag{1.21}$$

where $M_1^{-1}(v)$ is the function inverse to $M_1(u)$.

Thus, to represent $M(u)$ in the form of a composition, we must find an N-function $M_1(u)$ such that $M_2(u) = M[M_1^{-1}(|u|)]$ is also an N-function.

Since, for $u > 0$, we have that

$$p_2(u) = \frac{p[M_1^{-1}(u)]}{p_1[M_1^{-1}(u)]},$$

a necessary and sufficient condition that $M_2(u)$ be an N-function is that the function $p(u)/p_1(u)$ be non-decreasing, right-continuous, and satisfy conditions (1.13) because the continuous function $M_1^{-1}(v)$ is monotonic and tends to zero and to infinity together with v.

Thus, if we find a non-decreasing right-continuous function $p_1(u)$, satisfying conditions (1.13), such that the function $p(u)/p_1(u)$ is also non-decreasing, right-continuous, and satisfies conditions (1.13), then the functions

$$M_1(u) = \int_0^{|u|} p_1(t)\,dt$$

and $M_2(u)$, defined by equality (1.21), will be N-functions, where the equality $M(u) = M_2[M_1(u)]$ is valid.

In particular, the function $p_1(u)$ can be taken as the function

$$p_1(u) = [p(u)]^{\varepsilon_0} \quad (0 < \varepsilon_0 < 1).$$

We note further that if the N-function is the composition $M_2[M_1(u)]$ of two N-functions $M_1(u)$ and $M_2(u)$, then to each $k > 0$ there corresponds a constant $u_0 \geqslant 0$ such that, for $u \geqslant u_0$, we have $M(u) > M_2(ku)$.

§ 2. Complementary N-function

1. *Definition.* Let $p(t)$ be a function which is positive for $t > 0$, right-continuous for $t \geqslant 0$, non-decreasing, and satisfying conditions (1.13). We define the function $q(s)$ $(s \geqslant 0)$ by the equality

$$q(s) = \sup_{p(t) \leqslant s} t. \tag{2.1}$$

It is easily seen that the function $q(s)$ possesses the same properties as the function $p(t)$: it is positive for $s > 0$, right-continuous for $s \geqslant 0$, non-decreasing, and satisfies the conditions

$$q(0) = 0, \quad \lim_{s \to \infty} q(s) = \infty. \tag{2.2}$$

It follows directly from the definition of the function $q(s)$ that we have the inequalities

$$q[p(t)] \geqslant t, \quad p[q(s)] \geqslant s, \tag{2.3}$$

and, for $\varepsilon > 0$, that

$$q[p(t) - \varepsilon] \leqslant t, \quad p[q(s) - \varepsilon] \leqslant s. \tag{2.4}$$

If the function $p(t)$ is continuous and increases monotonically, then the function $q(s)$ is the ordinary inverse function of $p(t)$. In the general case, the function $q(s)$ is called the *right inverse* of $p(t)$. The function $p(t)$, in turn, is the right inverse of $q(s)$. Fig. 4 shows the graph of the function $q(s)$, the right inverse of the function $p(t)$, whose graph is shown in Fig. 2.

The functions

$$M(u) = \int_0^{|u|} p(t)\,dt, \quad N(v) = \int_0^{|v|} q(s)\,ds$$

are called *mutually complementary* N-functions.

Suppose $\Phi(u)$ and $\Psi(v)$ are mutually complementary N-functions. There are situations when we must consider the N-function $\Phi_1(u) = a\Phi(bu)$ $(a, b > 0)$. The N-function $\Psi_1(v)$ which is com-

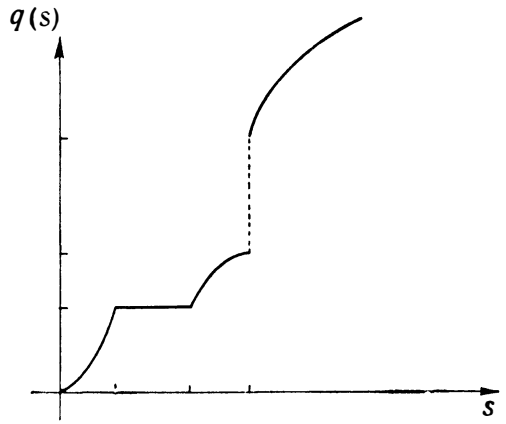

Fig. 4.

plementary to $\Phi_1(u)$ is defined by the equality

$$\Psi_1(v) = a\Psi\left(\frac{v}{ab}\right). \tag{2.5}$$

In fact, the right derivative $p_1(t)$ of the function $\Phi_1(u)$ equals $abp(bt)$, where $p(t)$ is the right derivative of the N-function $\Phi(u)$. From this it follows that $q_1(s) = (1/b)q(s/ab)$ and that

$$\Psi_1(v) = \int_0^{|v|} q_1(s)\,ds = \frac{1}{b}\int_0^{|v|} q\left(\frac{s}{ab}\right)ds = a\int_0^{|v|/ab} q(s)\,ds,$$

which implies (2.5).

2. *The Young inequality.* We shall use the line of reasoning usually utilized in the derivation of Hölder's inequality. In Fig. 5, the areas T and S express the values of the N-functions $M(u)$ and $N(v)$, respectively. It is geometrically clear that the following inequality holds: $uv \leqslant T + S = M(u) + N(v)$. In virtue of the fact that the functions $M(u)$ and $N(v)$ are even, the last inequality is valid for all u, v; it is called *Young's inequality*. Thus,

$$uv \leqslant M(u) + N(v). \tag{2.6}$$

It is clear, from Fig. 5 again, that inequality (2.6) reduces to an equality when $v = p(|u|)\operatorname{sgn} u$ if u is given and when $u = q(|v|)\operatorname{sgn} v$ if v is given. Thus,

$$|u|p(|u|) = M(u) + N[p(|u|)] \tag{2.7}$$

and

$$|v|q(|v|) = M[q(|v|)] + N(v). \tag{2.8}$$

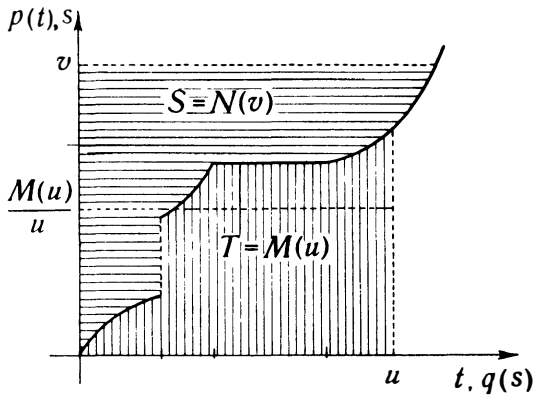

Fig. 5.

It follows from (2.6) that $N(v) \geqslant uv - M(u)$. In virtue of (2.8), this inequality goes over into an equality for $u = q(|v|)\operatorname{sgn} v$. Consequently,

$$N(v) = \max_{u \geqslant 0} [u|v| - M(u)]. \tag{2.9}$$

Formula (2.9) could have been taken as the definition of the N-function complementary to $M(u)$.

It follows from Young's inequality that $M^{-1}(v)N^{-1}(v) \leqslant 2v$ $(v > 0)$. On the other hand, it is clear from Fig. 5 that $N[M(u)/u] < M(u)$. It follows from this that, for $M(u) = v$, we have $v < M^{-1}(v)N^{-1}(v)$. Thus, for all $v > 0$, we have that

$$v < M^{-1}(v)N^{-1}(v) \leqslant 2v. \tag{2.10}$$

3. *Examples.* As we have already pointed out, the function $M_1(u) = |u|^\alpha/\alpha$ $(\alpha > 1)$ is an N-function. We shall compute the

complementary function to it. Clearly, for $t > 0$, we have that $p_1(t) = M_1'(t) = t^{\alpha-1}$. Therefore, $q_1(s) = s^{\beta-1}$ $(s \geqslant 0)$, where $1/\alpha + 1/\beta = 1$, and

$$N_1(v) = \int_0^{|v|} q_1(s)\,ds = \frac{|v|^\beta}{\beta}.$$

As a second example, we shall compute the N-function complementary to the N-function $M_2(u) = e^{|u|} - |u| - 1$. For this function, we have that $p_2(t) = M_2'(t) = e^t - 1$ $(t \geqslant 0)$, from which it follows that $q_2(s) = \ln(s+1)$ $(s \geqslant 0)$ and

$$N_2(v) = \int_0^{|v|} q_2(s)\,ds = (1 + |v|)\ln(1 + |v|) - |v|. \tag{2.11}$$

We note that it is impossible in many cases to find an explicit formula for the complementary N-function. For example, if $M(u) = e^{u^2} - 1$, then $p(t) = 2te^{t^2}$ and we cannot express $q(s)$ in the explicit form.

4. *Inequality for complementary functions.*

THEOREM 2.1. *Suppose that the inequality $M_1(u) \leqslant M_2(u)$ is satisfied for the N-functions $M_1(u)$ and $M_2(u)$ when $u \geqslant u_0$. Then the inequality $N_2(v) \leqslant N_1(v)$ holds for the complementary N-functions $N_1(v)$ and $N_2(v)$ when $v \geqslant v_0 = p_2(u_0)$.*

PROOF. Suppose $p_2(u)$ is the right derivative of the N-function $M_2(u)$. In virtue of the fact that the function $q_2(v)$ is monotone, the inequality $q_2(v) \geqslant u_0$ holds for $v \geqslant v_0 = p_2(u_0)$.

In virtue of (2.8), we have $q_2(v)\cdot v = M_2[q_2(v)] + N_2(v)$, and, in virtue of Young's inequality, we have

$$q_2(v)\cdot v \leqslant M_1[q_2(v)] + N_1(v),$$

so that

$$M_2[q_2(v)] + N_2(v) \leqslant M_1[q_2(v)] + N_1(v).$$

Since $M_2[q_2(v)] \geqslant M_1[q_2(v)]$ for $v \geqslant v_0$, we have $N_2(v) \leqslant N_1(v)$. ★

§ 3. The comparison of N-functions

1. *Definition.* In the sequel, an essential role will be played by the "rapidity of growth" of the values of an N-function as

$u \to \infty$. In this connection, it is convenient to introduce the following notation. We shall write

$$M_1(u) \prec M_2(u) \tag{3.1}$$

if there exist positive constants u_0 and k such that

$$M_1(u) \leqslant M_2(ku) \quad (u \geqslant u_0). \tag{3.2}$$

We shall say that the N-functions $M_1(u)$ and $M_2(u)$ are *comparable* if one of the relations $M_1(u) \prec M_2(u)$ or $M_2(u) \prec M_1(u)$ holds.

It is easily verified that $M_1(u) \prec M_2(u)$ and $M_2(u) \prec M_3(u)$ implies that $M_1(u) \prec M_3(u)$. A set of elements in which a relation of type (3.1), possessing the indicated property, is introduced, is called a *partially ordered* set. Thus, the N-functions form a set which is partially ordered relative to the symbol $\prec$.

The simplest example of N-functions, satisfying relation (3.1), are the functions $M_1(u) = |u|^{\alpha_1}$, $M_2(u) = |u|^{\alpha_2}$ $(\alpha_1, \alpha_2 > 1)$ for $\alpha_1 < \alpha_2$.

We now consider the N-function $M(u) = |u|^{\alpha}(|\ln |u|| + 1)$ $(\alpha > 1)$. It is clear that $|u|^{\alpha} \prec M(u) \prec |u|^{\alpha+\varepsilon}$ for arbitrary $\varepsilon > 0$.

2. *Equivalent N-functions.* We shall say that the N-functions $M_1(u)$ and $M_2(u)$ are *equivalent* and write $M_1(u) \sim M_2(u)$ if $M_1(u) \prec M_2(u)$ and $M_2(u) \prec M_1(u)$.

Clearly, every N-function is equivalent to itself and if two N-functions are equivalent to a third N-function, then they are equivalent. In virtue of this, the set of all N-functions is partitioned into classes of mutually equivalent functions.

It follows from the definition that the N-functions $M_1(u)$ and $M_2(u)$ are equivalent if, and only if, there exist positive constants k_1, k_2 and u_0 such that

$$M_1(k_1 u) \leqslant M_2(u) \leqslant M_1(k_2 u) \quad (u \geqslant u_0). \tag{3.3}$$

From these inequalities, it follows, in particular, that the N-function $M(u)$ is equivalent to the N-function $M(ku)$ for arbitrary $k > 0$. It is also clear that the N-functions $M(u)$ and $M_1(u)$ satisfying the condition

$$\lim_{u \to \infty} \frac{M(u)}{M_1(u)} = a > 0 \tag{3.4}$$

are equivalent.

THEOREM 3.1. *Suppose* $M_1(u) \prec M_2(u)$. *Then the corresponding complementary N-functions are connected by the relation* $N_2(v) \prec N_1(v)$.

PROOF. By hypothesis, $k, u_0 > 0$ can be found such that

$$M_1(u) \leqslant M_2(ku) \quad (u \geqslant u_0). \tag{3.5}$$

We set $M(u) = M_2(ku)$. The function $N(v)$, which is complementary to $M(u)$, equals $N_2(v/k)$ in virtue of (2.5).

Inequality (3.5) can be rewritten in the form $M_1(u) \leqslant M(u)$ $(u \geqslant u_0)$. In virtue of Theorem 2.1, there exists a $v_0 > 0$ such that $N(v) \leqslant N_1(v)$ $(v \geqslant v_0)$ from which it follows that $N_2(v) \leqslant N_1(kv)$ $(v \geqslant v_0/k)$. ★

The next theorem follows directly from Theorem 3.1.

THEOREM 3.2. *If the N-functions* $M_1(u)$ *and* $M_2(u)$ *are equivalent, then the N-functions complementary to them,* $N_1(v)$ *and* $N_2(v)$, *are also equivalent.*

Theorem 3.2 signifies that to a class of mutually equivalent N-functions there corresponds, in going over to the complementary functions, a class of N-functions which are also mutually equivalent.

3. *Principal part of an N-function.* A convex function $Q(u)$ will be called the *principal part* (p.p.) of the N-function $M(u)$ if $Q(u) = M(u)$ for large values of the argument.

THEOREM 3.3. *Suppose the convex function* $Q(u)$ *satisfies the condition*

$$\lim_{u\to\infty} \frac{Q(u)}{u} = \infty. \tag{3.6}$$

Then $Q(u)$ *is the principal part of some N-function* $M(u)$.

PROOF. It follows from condition (3.6) that $\lim_{u\to\infty} Q(u) = \infty$. We shall assume that $Q(u)$ is convex and positive for $u \geqslant u_0$. In virtue of Theorem 1.1, the function $Q(u)$ admits of the representation

$$Q(u) = \int_{u_0}^{u} p(t)\,dt + Q(u_0),$$

where $p(u)$ is a non-decreasing right-continuous function. This function satisfies the condition $\lim_{u\to\infty} p(u) = \infty$ inasmuch as the boundedness of the function $p(u)$, i.e. $p(u) \leqslant b$, would imply that

$Q(u) \leqslant b(u - u_0) + Q(u_0)$, which contradicts (3.6). Without loss of generality, we may assume that $p(u)$ is positive for $u \geqslant u_0$.

Since $p(u)$ increases indefinitely, there exists a $u_1 \geqslant u_0 + 1$ such that $p(u_1) > p(u_0 + 1) + Q(u_0)$. Then

$$Q(u_1) = \int_{u_0}^{u_0+1} p(t)\,dt + \int_{u_0+1}^{u_1} p(t)\,dt + Q(u_0) \leqslant$$

$$\leqslant p(u_0 + 1) + Q(u_0) + p(u_1)(u_1 - u_0 - 1) \leqslant p(u_1)(u_1 - u_0),$$

from which it follows that $\alpha = u_1 p(u_1)/Q(u_1) > 1$.

We define the function $M(u)$ by the equality

$$M(u) = \begin{cases} \dfrac{Q(u_1)}{u_1^\alpha}\,|u|^\alpha & \text{for} \quad |u| \leqslant u_1, \\ Q(u) & \text{for} \quad |u| \geqslant u_1. \end{cases}$$

The function $M(u)$ is an N-function inasmuch as its right derivative,

$$M'_+(u) = \begin{cases} \dfrac{\alpha Q(u_1)}{u_1^\alpha}\,u^{\alpha-1} & \text{for} \quad 0 \leqslant u \leqslant u_1, \\ p(u) & \text{for} \quad u \geqslant u_1, \end{cases}$$

is a function which is positive for $u > 0$, right-continuous for $u \geqslant 0$, non-decreasing, and such that it satisfies conditions (1.13). ★

4. *An equivalence criterion.* A set F on the real line will be called a *set of complete measure* if the set of points not belonging to F has measure zero.

We consider two N-functions,

$$M_1(u) = \int_0^{|u|} p_1(t)\,dt, \quad M_2(u) = \int_0^{|u|} p_2(t)\,dt. \tag{3.7}$$

LEMMA 3.1. *Suppose there exist constants k, $u_0 > 0$ and a set F of complete measure such that $p_1(u) \leqslant p_2(ku)$ $(u \geqslant u_0,\ u \in F)$. Then the N-functions*

$$M_1(u) = \int_0^{|u|} p_1(t)\,dt \quad \textit{and} \quad M_2(u) = \int_0^{|u|} p_2(t)\,dt$$

satisfy the relation $M_1(u) \prec M_2(u)$.

PROOF. Integrating the inequality, given in the hypothesis of the lemma, between the limits from u_0 to u, we obtain that

$$M_1(u) - M_1(u_0) \leqslant \frac{1}{k}[M_2(ku) - M_2(ku_0)] < \frac{1}{k} M_2(ku) \quad (u \geqslant u_0).$$

Without loss of generality, we may assume that $k > 1$. In virtue of the fact that $M_1(u)$ increases indefinitely, a $u_1 \geqslant u_0$ can be found such that, for $u \geqslant u_1$, we have

$$M_1(u) - M_1(u_0) \geqslant \frac{1}{k} M_1(u).$$

Therefore, for $u \geqslant u_1$, $M_1(u) \leqslant M_2(ku)$. ★

It follows from Lemma 3.1 that $M_1(u) \prec M_2(u)$ if the inequality $p_1[\alpha q_2(\beta u)] < u$ holds for large u.

LEMMA 3.2. *Let*

$$\lim_{\substack{u\to\infty \\ u\in F}} \frac{p_1(u)}{p_2(u)} = b > 0, \tag{3.8}$$

where F is a set of complete measure. Then $M_1(u) \sim M_2(u)$.

PROOF. In virtue of (3.8), there exists a $u_0 > 0$ such that for $u \geqslant u_0$, $u \in F$, we have $p_1(u) \leqslant 2bp_2(u)$. Integrating the last inequality between the limits from u_0 to u, we obtain

$$M_1(u) - M_1(u_0) \leqslant 2b[M_2(u) - M_2(u_0)] \quad (u \geqslant u_0),$$

from which it follows, in virtue of the fact that $\lim_{u\to\infty} M_2(u) = \infty$, that $M_1(u) \leqslant (2b + 1)M_2(u)$ for large values of u. It follows from this inequality, in virtue of (1.7), that for large values of u, $M_1(u) \leqslant M_2[(2b + 1)u]$, i.e. that $M_1(u) \prec M_2(u)$.

The relation $M_2(u) \prec M_1(u)$ is proved analogously. ★

Only the values of the functions $p_1(u)$ and $p_2(u)$ for large values of the argument play a role in the conditions of Lemma 3.2. Here, as also in a number of other cases, when considering the right derivatives $p(u)$ of the N-function $M(u)$, it is important to have a formula for the function $p(u)$ only for large values of u. In this connection, we shall make use of the following definition: a function $\varphi(u)$ is called the *principal part* (p.p.) of the function $p(u)$ if $\varphi(u)$ and $p(u)$ coincide for large values of the argument.

THEOREM 3.4. *Suppose the N-functions* (3.7) *and the N-functions*

$$N_1(v) = \int_0^{|v|} q_1(s)\,ds, \quad N_2(v) = \int_0^{|v|} q_2(s)\,ds,$$

complementary to them, are given. Assume there exists a set F_1 *of complete measure such that*

$$\lim_{\substack{v\to\infty \\ v\in F_1}} \frac{p_1[q_2(v)]}{v} = b > 0. \tag{3.9}$$

Then $M_1(v) \sim M_2(v)$.

PROOF. We introduce the notation $q_2(v) = u$. In virtue of (2.3),

$$p_2[q_2(v)] = p_2(u) \geqslant v \tag{3.10}$$

and, in virtue of (2.4),

$$p_2(u - \varepsilon) \leqslant v \tag{3.11}$$

for arbitrary $\varepsilon > 0$.

We denote by F the subset of F_1 consisting of those points at which the functions $p_1(u)$ and $p_2(u)$ are continuous. Since every monotonic function has at most a denumerable number of points of discontinuity, we have that F is also a set of complete measure.

It follows from (3.10) that

$$\frac{p_1(u)}{p_2(u)} \leqslant \frac{p_1(u)}{v} = \frac{p_1[q_2(v)]}{v},$$

from which it follows, in virtue of (3.9), that

$$\overline{\lim_{\substack{u\to\infty \\ u\in F_1}}} \frac{p_1(u)}{p_2(u)} \leqslant b. \tag{3.12}$$

It follows from (3.11) that for all $u \in F$,

$$\frac{p_1(u)}{p_2(u)} = \lim_{\varepsilon\to 0} \frac{p_1(u)}{p_2(u-\varepsilon)} \geqslant \frac{p_1(u)}{v} = \frac{p_1[q_2(v)]}{v}$$

from which it follows, in virtue of (3.9), that

$$\lim_{\substack{u\to\infty \\ u\in F}} \frac{p_1(u)}{p_2(u)} \geqslant b. \tag{3.13}$$

It follows from inequalities (3.12) and (3.13) that

$$\lim_{\substack{u\to\infty \\ u\in F}} \frac{p_1(u)}{p_2(u)} = b.$$

The last inequality and Lemma 3.2 imply that $M_1(u) \sim M_2(u)$. ★

5. *The existence of various classes.* In connection with the introduction of classes of equivalent N-functions, there arises the question of "how many" distinct classes of this sort are there? It is clear, for example, that the N-functions $|u|^\alpha$ belong to distinct classes for distinct $\alpha > 1$. The N-function $M(u) = (1 + |u|) \ln (1 + |u|) - |u|$ satisfies the relation $M(u) \prec |u|^\alpha$ $(\alpha > 1)$, but it is not equivalent to any of the N-functions $|u|^\alpha$. Nor is the N-function $M_1(u) = e^{|u|} - |u| - 1$, which satisfies the relation $|u|^\alpha \prec M_1(u)$, equivalent to any of the N-functions $|u|^\alpha$.

Suppose, now, that

$$M_n(u) = \int_0^{|u|} p_n(t)\,dt \quad (n = 1, 2, \cdots) \tag{3.14}$$

is an arbitrary sequence of N-functions. We construct N-functions $M(u)$ and $\Phi(u)$ such that

$$M_n(u) \prec M(u) \quad (n = 1, 2, \cdots) \tag{3.15}$$

and

$$\Phi(u) \prec M_n(u) \quad (n = 1, 2, \cdots). \tag{3.16}$$

Let $p(t) = p_1(t) + p_2(t) + \cdots + p_n(t)$ for $n - 1 \leqslant t < n$. The function $p(t)$ is right-continuous, monotonically increasing, and it satisfies conditions (1.13). In virtue of Lemma 3.1, the N-function

$$M(u) = \int_0^{|u|} p(t)\,dt$$

will satisfy relation (3.15).

By what we have already proved, an N-function $\Psi(v)$ can be constructed satisfying the relations $N_n(v) \prec \Psi(v)$, where the $N_n(v)$ are the complementary functions to the N-functions (3.14). In virtue of Theorem 3.1, the complementary N-function $\Phi(u)$ to $\Psi(v)$ will satisfy conditions (3.16).

Suppose $M(u)$ is an N-function. The function $M_1(u) = e^{M(u)} - 1$ is also an N-function. Clearly, $M(u) \prec M_1(u)$. It is clear that in the case when $M(u)$ does not increase faster than a power function, $M_1(u)$ is not equivalent to $M(u)$. These functions are also not mutually equivalent for many other N-functions. However, there also exist N-functions $M(u)$ such that $e^{M(u)} - 1 \sim M(u)$ (we leave it to the reader to construct an example!).

For an arbitrary N-function $M(u)$, it is not difficult to construct N-functions $Q(u)$ and $R(u)$, which are not equivalent to $M(u)$, such that $Q(u) \prec M(u) \prec R(u)$. For this purpose, it is sufficient to define the right derivative $r(u)$ of the function $R(u)$ by the equality $r(u) = np(nu)$ for $n - 1 \leqslant u < n$ $(n = 1, 2, \cdots)$. The function $Q(u)$ can be defined as the complementary function to the N-function $\Psi(v)$ which satisfies the conditions: $N(v) \prec \Psi(v)$ and $\Psi(v)$ is not equivalent to $N(v)$, where $N(v)$ is the complementary function to $M(u)$.

It is easily seen that the functions $Q(u)$ and $R(u)$ thus constructed possess the following property: to each $n = 1, 2, \cdots$, there exists a u_n^* such that, for $u > u_n^*$, we have

$$Q(u) < M\left(\frac{u}{n}\right) < M(nu) < R(u). \tag{3.17}$$

In concluding this section, we shall show that to each N-function $M(u)$ there corresponds an N-function $\Phi(u)$ such that neither the relation $M(u) \prec \Phi(u)$ nor the relation $\Phi(u) \prec M(u)$ holds. To this end, we first construct the N-functions $Q(u)$ and $R(u)$ which satisfy relations (3.17). Without loss of generality, we may assume that $Q(u) < R(u)$ $(u \geqslant u_0)$, where u_0 is a positive number. We shall now describe how the graph of the function $\Phi(u)$ is constructed (see Fig. 6).

We first set $\Phi(u) = Q(u)$ for $0 \leqslant u \leqslant u_0$. Next, we draw a straight line through the points with the coordinates $\{u_0, Q(u_0)\}$ and $\{u_0 + 1, R(u_0 + 1)\}$. In virtue of property (1.16), this straight line intersects the graph of the function $Q(u)$ is still one more point whose abscissa will be denoted by u_1. Through the points $\{u_1, Q(u_1)\}$ and $\{u_1 + 1, R(u_1 + 1)\}$ we draw a new straight line until it intersects the graph of the function $Q(u)$; the abscissa of the new point will be denoted by u_2. Continuing this process, we obtain a polygonal arc connecting the points with the coordinates

$\{u_0, Q(u_0)\}$, $\{u_1, Q(u_1)\}$, $\{u_2, Q(u_2)\}$, and so forth. This polygonal arc will then be the graph of the N-function $\Phi(u)$ for $u \geqslant u_0$.

By construction, $\Phi(u)$ possesses the following properties:

$$\Phi(u_n) = Q(u_n) \quad (n = 1, 2, \cdots) \tag{3.18}$$

and

$$\Phi(u_n + 1) = R(u_n + 1) \quad (n = 1, 2, \cdots). \tag{3.19}$$

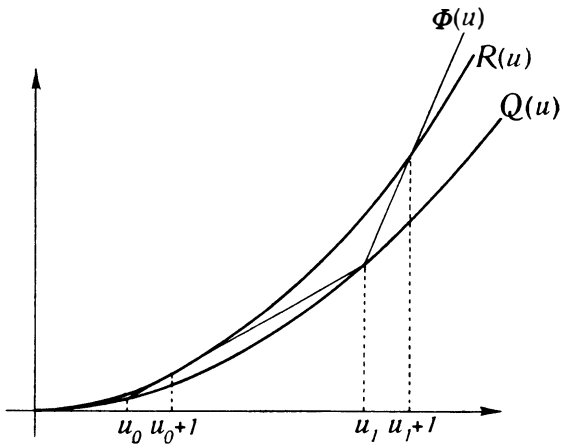

Fig. 6.

We now assume that $\Phi(u) \prec M(u)$. Then k and $u^* > 0$ can be found such that

$$\Phi(u) \leqslant M(ku) \quad (u \geqslant u^*). \tag{3.20}$$

In virtue of (3.17), a $u_n > u^*$ can be found such that

$$M[k(u_n + 1)] < R(u_n + 1).$$

Then, in virtue of (3.19),

$$M[k(u_n + 1)] < \Phi(u_n + 1),$$

which contradicts (3.20).

It is proved analogously that the relation $M(u) \prec \Phi(u)$ does not hold.

We leave it to the reader to prove that for an arbitrary sequence of N-functions $M_n(u)$ $(n = 1, 2, \cdots)$ there exist N-functions $\Phi(u)$ and $\Psi(u)$ such that $\Phi(u) \prec M_n(u) \prec \Psi(u)$, where $\Phi(u)$ and $\Psi(u)$ are equivalent to none of the functions $M_n(u)$.

§ 4. The Δ_2-condition

1. *Definition.* We say that the N-function $M(u)$ *satisfies the* Δ_2*-condition* for large values of u if there exist constants $k > 0$, $u_0 \geqslant 0$ such that

$$M(2u) \leqslant kM(u) \quad (u \geqslant u_0). \tag{4.1}$$

It is easily seen that we always have $k > 2$ inasmuch as, in virtue of (1.15), $M(2u) > 2M(u)$ for $u \neq 0$.

The Δ_2-condition is equivalent to the satisfaction of the inequality

$$M(lu) \leqslant k(l)M(u) \tag{4.2}$$

for large values of u, where l can be any number larger than unity.

In fact, let $2^n \geqslant l$. Then it follows from (4.1), with $u \geqslant u_0$, that

$$M(lu) \leqslant M(2^n u) \leqslant k^n M(u) = k(l)M(u).$$

Conversely, if $2 \leqslant l^n$, then it follows from (4.2) that

$$M(2u) \leqslant M(l^n u) \leqslant k^n(l)M(u).$$

The N-functions $M(u) = a|u|^\alpha$ $(\alpha > 1)$ can serve as a simple example of functions satisfying the Δ_2-condition for all values of u inasmuch as $M(2u) = a2^\alpha |u|^\alpha = 2^\alpha M(u)$.

Clearly, the Δ_2-condition is satisfied for large values of u if

$$\overline{\lim_{u\to\infty}} \frac{}{M(u)} < \infty. \tag{4.3}$$

It is also easily seen that the fulfillment of the Δ_2-condition for all u, i.e. the satisfaction of inequality (4.1) for all $u \geqslant 0$, is equivalent to condition (4.3) and the condition

$$\overline{\lim_{u\to 0}} \frac{M(2u)}{M(u)} < \infty. \tag{4.4}$$

If $M(u)$ satisfies the Δ_2-condition, then any N-function which is equivalent to $M(u)$ also satisfies this condition. In fact, suppose $M_1(u) \sim M(u)$. This means that numbers $\alpha < \beta$ and $u_1 \geqslant 0$ can be found such that $M(\alpha u) \leqslant M_1(u) \leqslant M(\beta u)$ $(u \geqslant u_1)$. Consequently, for $u \geqslant \max\{u_0, u_1\}$, we have that

$$M_1(2u) \leqslant M(2\beta u) \leqslant k(2\beta/\alpha)M(\alpha u) \leqslant k(2\beta/\alpha)M_1(u).$$

We note that in every class of equivalent N-functions which satisfy the Δ_2-condition, there are N-functions which satisfy inequality (4.1) for all u. In fact, suppose $M(u)$ satisfies inequality (4.1) for $u \geqslant u_0$. As in the proof of Theorem 3.3, we define the N-function $M_1(u)$ by the equality

$$M_1(u) = \begin{cases} \dfrac{M(u_0)}{u_0^\alpha} |u|^\alpha & \text{for} \quad |u| \leqslant u_0, \\ M(u) & \text{for} \quad |u| \geqslant u_0, \end{cases} \tag{4.5}$$

where $\alpha = u_0 p(u_0)/M(u_0) > 1$. Then, for all u, we have

$$M_1(2u) \leqslant \max\{2^\alpha, k\} M_1(u).$$

2. *Tests for the Δ_2-condition.*

THEOREM 4.1. *A necessary and sufficient condition that the N-function $M(u)$ satisfy the Δ_2-condition is that there exist constants α and $u_0 > 0$ such that, for $u \geqslant u_0$,*

$$\frac{up(u)}{M(u)} < \alpha, \tag{4.6}$$

where $p(u)$ is the right derivative of the N-function $M(u)$.

PROOF. Inasmuch as $up(u)$ is always $> M(u)$, $\alpha > 1$. Suppose $u \geqslant u_0$. Then from (4.6) we obtain that

$$\int_u^{2u} \frac{p(t)}{M(t)}\, dt < \alpha \int_u^{2u} \frac{dt}{t} = \alpha \ln 2$$

or, equivalently, that $M(2u) < 2^\alpha M(u)$. The sufficiency of the condition (4.6) is thus proved.

Now, let $M(2u) \leqslant kM(u)$ for $u \geqslant u_0$. Then

$$kM(u) \geqslant M(2u) = \int_0^{2u} p(t)\, dt > \int_u^{2u} p(t)\, dt > up(u),$$

i.e. inequality (4.6) is satisfied for $u \geqslant u_0$. ★

It is clear from the proof that $M(u)$ satisfies the Δ_2-condition for all $u > 0$ if inequality (4.6) is satisfied for all $u > 0$.

Theorem 4.1 enables us to prove quite simply that N-functions $M(u)$ which satisfy the Δ_2-condition do not increase more rapidly

than exponential functions. In fact, if the Δ_2-condition is satisfied, then it follows from (4.6) that

$$\int_{u_0}^{u} \frac{p(t)}{M(t)}\,dt < \alpha \int_{u_0}^{u} \frac{dt}{t},$$

i.e. that

$$M(u) < \frac{M(u_0)}{u_0^{\alpha}}\, u^{\alpha} \tag{4.7}$$

for $u \geqslant u_0$.

It is easily verified that an N-function $M(u)$ satisfies the Δ_2-condition if its right derivative $p(u)$ satisfies the inequality

$$p(2u) \leqslant lp(u) \quad (u \geqslant u_0), \tag{4.8}$$

where $l > 1$, $u_0 \geqslant 0$.

Inequality (4.8) is satisfied, in particular, if the function $p(u)$ is convex downward for large values of the argument, i.e.

$$p\left(\frac{u_1 + u_2}{2}\right) \geqslant \frac{p(u_1) + p(u_2)}{2}$$

for large u_1 and u_2.

3. *The Δ_2-condition for the complement to an N-function.* We shall frequently be interested in the following question: Given an N-function $N(v)$, how can one determine directly whether or not the N-function $M(u)$ complementary to it satisfies the Δ_2-condition?

THEOREM 4.2. *A necessary and sufficient condition that the complementary function $M(u)$ to the N-function $N(v)$ satisfy the Δ_2-condition is that there exist constants $l > 1$ and $v_0 \geqslant 0$ such that*

$$N(v) \leqslant \frac{1}{2l} N(lv) \quad (v \geqslant v_0). \tag{4.9}$$

PROOF. Assume that condition (4.9) is satisfied. We set $N_1(v) = \{1/(2l)\}N(lv)$. In virtue of equality (2.5), the complementary N-function $M_1(u)$ to $N_1(v)$ is defined by the equality $M_1(u) = \{1/(2l)\}M(2u)$. Inequality (4.9) can be written in the form $N(v) \leqslant N_1(v)$. It follows from this, in virtue of Theorem 2.1, that

$M_1(u) \leqslant M(u)$ or, equivalently, that $M(2u) \leqslant 2lM(u)$ for large values of the argument.

It is proved analogously that (4.1) implies (4.9). ★

If (4.9) is satisfied for all $v > 0$, then $M(u)$ also satisfies the Δ_2-condition for all u.

As was noted in the preceding subsection, an N-function satisfies the Δ_2-condition if its derivative is convex downward for large values of the argument. Manifestly, a function is convex downward if the function inverse to it is convex. Thus, the N-function $M(u)$ satisfies the Δ_2-condition if the N-function $N(v)$ complementary to it has a convex derivative.

To prove the next theorem, we shall need the following auxiliary assertion.

LEMMA 4.1. *Suppose the functions $p(u)$ and $q(v)$ are continuous. Then, a necessary and sufficient condition that inequalities* (4.6) *be satisfied is that the inequality*

$$\frac{vq(v)}{N(v)} > \frac{\alpha}{\alpha - 1} \tag{4.10}$$

hold for large values of v.

PROOF. We shall show, for example, that (4.6) implies (4.10). In virtue of (2.7), we have that $M(u) = up(u) - N[p(u)]$. Therefore, it follows from (4.6) that

$$\frac{up(u)}{up(u) - N[p(u)]} < \alpha \quad (u \geqslant u_0),$$

from which it follows that

$$\frac{up(u)}{N[p(u)]} > \frac{\alpha}{\alpha - 1} \quad (u \geqslant u_0). \tag{4.11}$$

If we set $u = q(v)$ in this inequality (in virtue of the continuity of the functions $p(u)$ and $q(v)$ we have $p(u) = v$), then we obtain (4.10). It is proved analogously that (4.6) follows from (4.10). ★

Now, the lemma just proved and Theorem 4.1 imply Theorem 4.3.

THEOREM 4.3. *Suppose the N-function $N(v)$ has, for large values of v, a monotonically increasing continuous derivative. Then the N-function $M(u)$ complementary to it satisfies the Δ_2-condition if,*

and only if, the inequality

$$\frac{vq(v)}{N(v)} > \alpha_1 \tag{4.12}$$

holds for large values of v, *where* $\alpha_1 > 1$.

To prove this theorem, it suffices to note that the continuity of the function $p(u)$ follows from the monotonicity of the function $q(v)$.

As in the case of Theorem 4.1, the N-function $M(u)$ satisfies the Δ_2-condition for all u if inequality (4.12) holds for all $u > 0$.

4. *Examples.* As we have already remarked, the N-functions $M(u) = a|u|^\alpha$ $(\alpha > 1)$ satisfy the Δ_2-condition for all values of u.

As the next example, we consider the N-function

$$M(u) = |u|^\alpha(|\ln |u|| + 1). \tag{4.13}$$

For this function, we have

$$\frac{up(u)}{M(u)} = \frac{\alpha + \alpha \ln u + 1}{\ln u + 1}$$

when $u > 1$, from which it follows that

$$\lim_{u\to\infty} \frac{up(u)}{M(u)} = \alpha.$$

Therefore, in virtue of Theorem 4.1, the N-function (4.13) satisfies the Δ_2-condition for large values of u. A direct calculation shows that

$$\lim_{u\to\infty} \frac{M(2u)}{M(u)} = \lim_{u\to 0} \frac{M(2u)}{M(u)} = 2^\alpha.$$

This means that conditions (4.3) and (4.4) are satisfied. This signifies that the N-function (4.13) satisfies the Δ_2-condition for all u.

We shall leave it to the reader to show that the N-function complementary to the N-function (4.13) also satisfies the Δ_2-condition.

The N-function

$$N(v) = e^{|v|} - |v| - 1 \tag{4.14}$$

does not satisfy the Δ_2-condition inasmuch as it increases more

rapidly than any exponential function. The derivative of the function $N(v)$ is $e^v - 1$ $(v \geqslant 0)$ — it is convex. It follows from the remark made in the preceding subsection that the function $M(u)$ complementary to $N(v)$ satisfies the Δ_2-condition. It would not be difficult to verify that the function $N(v)$ satisfies condition (4.9). The fact that the function $M(u)$ complementary to the N-function (4.14) satisfies the Δ_2-condition can be verified directly since its expression in explicit form is known (see (2.10)): $M(u) = (1 + |u|) \ln (1 + |u|) - |u|$. In this connection, it is easily seen that the Δ_2-condition is satisfied for all u.

We now consider the N-function

$$N(v) = e^{v^2} - 1, \tag{4.15}$$

for which one cannot find the complementary function $M(u)$ in the explicit form. However, one can also show for it that $M(u)$ satisfies the Δ_2-condition for all values of the argument. To do this, we use Theorem 4.2.

We note first of all that the function $\varphi(t) = e^{4t} - 4e^t + 3$ increases monotonically for $t > 0$ since $\varphi'(t) = 4e^t(e^{3t} - 1) > 0$. Consequently, for $v > 0$, we have that $(e^{4v^2} - 1)/4 > e^{v^2} - 1$. The last inequality is condition (4.9) for the function (4.15) for $l = 2$.

When considering the preceding examples, the conjecture could have been made that at least one of the two mutually complementary N-functions satisfies the Δ_2-condition. Moreover, the conjecture could have been made that every N-function which increases less rapidly than a power function satisfies the Δ_2-condition. We shall introduce an example which shows that both these conjectures are incorrect.

We shall construct an N-function $M(u)$ by giving its derivative $p(t)$ by the equality

$$p(t) = \begin{cases} t & \text{if } t \in [0, 1), \\ k! & \text{if } t \in [(k-1)!, k!) \quad (k = 2, 3, \cdots). \end{cases}$$

In order to prove that the N-function $M(u) = \int_0^{|u|} p(t)\,dt$ does not satisfy the Δ_2-condition it suffices to show that there exists

a sequence of numbers $u_n \to \infty$ such that

$$M(2u_n) > nM(u_n) \quad (n = 1, 2, \cdots). \tag{4.16}$$

Let $u_n = n!$ $(n = 1, 2, \cdots)$. Then

$$M(2u_n) > \int_{n!}^{2n!} p(t)\,dt > (n+1)!\cdot n!,$$

and

$$nM(u_n) = n \int_0^{n!} p(t)\,dt < n\cdot n!\cdot n!,$$

from which (4.16) follows.

It is clear that the function $q(s)$ is defined by the equality

$$q(s) = \begin{cases} s & \text{if } s \in [0, 1), \\ (k-1)! & \text{if } s \in [(k-1)!, k!) \quad (k = 2, 3, \cdots). \end{cases}$$

We shall show that the N-function $N(v) = \int_0^{|v|} q(s)\,ds$ does not satisfy the Δ_2-condition either. To this end, we consider the sequence of numbers $v_n = n!$ $(n = 1, 2, \cdots)$. Then

$$N(2v_n) > \int_{n!}^{2n!} q(s)\,ds > n!\cdot n!,$$

and

$$nN(v_n) = n \int_0^{n!} q(s)\,ds < n\cdot n!\cdot(n-1)! = n!\cdot n!,$$

i.e. $N(2v_n) > nN(v_n)$ $(n = 1, 2, \cdots)$. In this connection, the function $N(v)$ increases no more rapidly than $v^2/2$ since $q(s) \leqslant s$ $(s \geqslant 0)$.

§ 5. The Δ'-condition

1. *Definition.* We say that the N-function $M(u)$ *satisfies the* Δ'*-condition* if there exist positive constants c and u_0 such that

$$M(uv) \leqslant cM(u)M(v) \quad (u, v \geqslant u_0). \tag{5.1}$$

LEMMA 5.1. *If the N-function $M(u)$ satisfies the Δ'-condition, then it also satisfies the Δ_2-condition.*

PROOF. Let $k = cM(u_0 + 2)$. Then for $u \geqslant u_0 + 2$ we have

$$M(2u) \leqslant M[(u_0 + 2)u] \leqslant cM(u_0 + 2)M(u) = kM(u). \bigstar$$

Suppose the N-function $M(u)$ satisfies the Δ'-condition and that the N-function $M_1(u)$ is equivalent to $M(u)$. We shall show that then $M_1(u)$ also satisfies the Δ'-condition, i.e. that satisfaction of the Δ'-condition is a property of a class of mutually equivalent N-functions. Since $M(u) \sim M_1(u)$, there exist positive constants k_1, k_2 and u_1 such that

$$M(k_1 u) \leqslant M_1(u) \leqslant M(k_2 u) \quad (u \geqslant u_1). \tag{5.2}$$

It is convenient to assume that $k_1 < 1$, u_0, u_1, $k_2 > 1$.

In virtue of Lemma 5.1, $k_3 > 0$ and $u_2 \geqslant 0$ can be found such that

$$M\left(\frac{\sqrt{k_2}}{k_1} u\right) \leqslant k_3 M(u) \quad (u \geqslant u_2). \tag{5.3}$$

Consequently, for u, $v \geqslant \max\{u_0, u_1, u_2/k_1\}$, we have

$$M_1(uv) \leqslant M(k_2 uv) < cM(\sqrt{k_2}u)M(\sqrt{k_2}v) \leqslant$$
$$\leqslant ck_3^2 M(k_1 u)M(k_1 v) \leqslant ck_3^2 M_1(u)M_1(v).$$

We do not know whether or not there exists in each class of equivalent N-functions, satisfying the Δ'-condition, a function which satisfies this condition for all u, v.

It is necessary to note that the class of N-functions which satisfy the Δ'-condition is already essentially the class of N-functions satisfying the Δ_2-condition. We consider, for example, the function $M(u) = u^2/\ln(e + |u|)$. It is an N-function inasmuch as its derivative $p(u) = \{2u(u + e)\ln(u + e) - u^2\}/\{(u + e)\ln^2(u + e)\}$ $(u \geqslant 0)$ satisfies conditions (1.13) and increases monotonically. Only the last assertion is needed in the proof; this follows from the fact that

$$p'(u) = \frac{2}{(u + e)^2 \ln^3(u + e)} \left[(u + e)^2 \ln^2(u + e) - \right.$$
$$\left. - 2u(u + e)\ln(u + e) + u^2 + \frac{u^2 \ln(u + e)}{2}\right] >$$
$$> \frac{2}{(u + e)^2 \ln^3(u + e)} [(u + e)\ln(u + e) - u]^2 \geqslant 0 \quad (u > 0).$$

The N-function $M(u)$ satisfies the Δ_2-condition since

$$\lim_{u\to\infty} \frac{M(2u)}{M(u)} = 4.$$

This function does not satisfy the Δ'-condition since

$$\lim_{u\to\infty} \frac{M(u^2)}{M^2(u)} = \infty.$$

2. *Sufficiency criteria for the satisfaction of the Δ'-condition.*

THEOREM 5.1. *Suppose there exists a number $u_0 > 1$ such that for every fixed $u \geqslant u_0$ the function $h(t) = p(ut)/p(t)$ does not increase for $t \geqslant u_0$. Then the N-function*

$$M(u) = \int_0^{|u|} p(t)\,dt$$

satisfies the Δ'-condition.

PROOF. Let $u, v \geqslant u_0$. Then, by the condition of the theorem, we have that

$$\frac{p(ut)}{p(t)} \leqslant \frac{p(uu_0)}{p(u_0)} \qquad (t \geqslant u_0).$$

Using this inequality in the expressions

$$M(uv) = \int_0^{uv} p(t)\,dt = u\int_0^{v} p(ut)\,dt = u\int_0^{u_0} p(ut)\,dt + u\int_{u_0}^{u} p(ut)\,dt,$$

we obtain that

$$M(uv) \leqslant uu_0p(uu_0) + u\,\frac{p(uu_0)}{p(u_0)}\int_0^{v} p(t)\,dt =$$

$$= uu_0p(uu_0)\left[1 + \frac{M(v)}{u_0p(u_0)}\right]$$

and, since

$$uu_0p(uu_0) < \frac{1}{u_0 - 1}\int_{uu_0}^{uu_0^2} p(t)\,dt \leqslant \frac{1}{u_0 - 1}\int_0^{uu_0^2} p(t)\,dt = \frac{M(uu_0^2)}{u_0 - 1},$$

that

$$M(uv) \leqslant \frac{M(uu_0^2)}{u_0 - 1}\left[1 + \frac{M(v)}{u_0 p(u_0)}\right]. \tag{5.4}$$

It follows from the last inequality that the N-function $M(u)$ satisfies the Δ_2-condition. In fact, it follows from (5.4), for $u = u_0$, that

$$M(u_0 v) \leqslant \frac{2M(u_0^3)}{(u_0 - 1)u_0 p(u_0)} M(v)$$

for $v \geqslant v_0$, where v_0 is a number such that $M(v_0) > u_0 p(u_0)$.

In virtue of the Δ_2-condition, a $k > 0$ can be found such that $M(uu_0^2) \leqslant kM(u)$ for $u \geqslant v_0$. Hence, it follows from (5.4) that

$$M(uv) \leqslant \frac{2k}{(u_0 - 1)u_0 p(u_0)} M(u)M(v) \quad (u, v \geqslant v_0). \bigstar$$

We shall now assume that the function $p(t)$ is differentiable for large values of t.

LEMMA 5.2. *The function $h(t) = p(ut)/p(t)$ does not increase for fixed $u \geqslant u_0 > 1$ for $t \geqslant u_0$ if the function $g(t) = tp'(t)/p(t)$ does not increase for $t \geqslant u_0$.*

PROOF. The function $h(t)$ does not increase for $t \geqslant u_0$ if, and only if, its derivative

$$h'(t) = \frac{up'(ut)p(t) - p'(t)p(ut)}{p^2(t)}$$

is not positive for $t \geqslant u_0$, i.e. if, and only if,

$$\frac{up'(ut)}{p(ut)} \leqslant \frac{p'(t)}{p(t)}.$$

The last inequality follows directly from the condition of the lemma. $\bigstar$

This lemma and Theorem 5.1 imply the next theorem.

THEOREM 5.2. *Suppose that for large values of t the function $p(t)$ is differentiable and that the function*

$$g(t) = \frac{tp'(t)}{p(t)} \tag{5.5}$$

does not increase. Then the N-function

$$M(u) = \int_0^{|u|} p(t)\,dt$$

satisfies the Δ'-condition.

3. *The Δ'-condition for the complementary function.*

THEOREM 5.3. *Suppose the derivative $p(u)$ of the N-function $M(u)$ is differentiable for $u \geqslant u_0 > 1$ and that the function $g(t) = tp'(t)/p(t)$ does not decrease for $t \geqslant u_0$. Then the N-function*

$$N(v) = \int_0^{|v|} q(s)\,ds,$$

which is complementary to the N-function $M(u)$, satisfies the Δ'-condition.

PROOF. Since the function $g(t)$ does not decrease, for sufficiently large t it assumes positive values. This means that for large values of the argument, we have $p'(t) > 0$, from which the differentiability of the function $q(s)$, the inverse of the function $p(t)$, follows.

In view of Lemma 5.2, to prove the theorem, it suffices to show that an $s_0 > 1$ can be found such that the function $g_1(s) = sq'(s)/q(s)$ does not increase for $s \geqslant s_0$.

We set $s = p(t)$. Then, for $t > t_0 = \max\{u_0, q(1)\}$, $s > s_0 = p(t_0) > 1$, we have $q'(s) = 1/p'(t)$,

$$g_1(s) = \frac{sq'(s)}{q(s)} = \frac{p(t)}{tp'(t)} = \frac{1}{g(t)},$$

and, since the function $g(t)$ does not decrease, the function $g_1(s)$ does not increase. ★

In the next subsection, we shall isolate a special class of N-functions whose complementary functions satisfy the Δ'-condition.

4. *Examples.* If $M_1(u) = |u|^\alpha/\alpha$ $(\alpha > 1)$, then, obviously, $M_1(uv) = \alpha M_1(u)M_1(v)$ for all u, v, i.e. $M_1(u)$ satisfies the Δ'-condition.

A second example of an N-function which satisfies the Δ'-condition for all u, v is given by the N-function

$$M_2(u) = |u|^\alpha(|\ln|u|| + 1) \quad (\alpha > 1).$$

In fact,

$$M_2(uv) = |uv|^\alpha(|\ln |uv|| + 1) \leqslant$$
$$\leqslant |u|^\alpha |v|^\alpha(|\ln |u|| + |\ln |v|| + 1) \leqslant$$
$$\leqslant |u|^\alpha(|\ln |u|| + 1)\cdot|v|^\alpha(|\ln |v|| + 1) = M_2(u)M_2(v).$$

We now consider the N-function

$$M_3(u) = (1 + |u|) \ln (1 + |u|) - |u|.$$

The function $Q(u) = u \ln u$, which is convex for large values of u, satisfies condition (3.6). In virtue of Theorem 3.3, the function $Q(u)$ is the principal part of some N-function $\Phi(u)$: p.p. $\Phi(u) = Q(u)$.

The function $\Phi(u)$ satisfies the Δ'-condition in virtue of Theorem 5.5 since for this function we have that

$$g(t) = \frac{\ln t + 1}{\ln t} = 1 + \frac{1}{\ln t}.$$

The N-functions $\Phi(u)$ and $M_3(u)$ satisfy condition (3.4) and, consequently, they are equivalent. This means that the N-function $M_3(u)$ also satisfies the Δ'-condition.

It turns out that the N-function $M_3(u)$ does not satisfy the Δ'-condition for all u, v. In fact, if there existed a constant c such that $M(uv) \leqslant cM(u)M(v)$ for all u, v, then the values of the function

$$f(u, v) = \frac{M(u)M(v)}{M(uv)} =$$
$$= \frac{[(1 + |u|) \ln (1 + |u|) - |u|][(1 + |v|) \ln (1 + |v|) - |v|]}{(1 + |uv|) \ln (1 + |uv|) - |uv|}$$

would be bounded below by the positive number $1/c$. But, if we set $u = n$, $v = 1/\sqrt{n}$, then it is easily verified that

$$\lim_{n\to\infty} f\left(n, \frac{1}{\sqrt{n}}\right) = 0.$$

As the last example, we shall show that the Δ'-condition is satisfied by the N-function $N(v)$ which is complementary to the N-function

$$M(u) = (1 + |u|)^{\sqrt{\ln(1+|u|)}} - 1,$$

which can be verified by making use of Theorem 5.3. For $M(u)$,

the function

$$g(t) = \frac{tp'(t)}{p(t)} = \left[\tfrac{3}{2}\sqrt{\ln(1+t)} - 1 + \frac{1}{2\ln(1+t)}\right]\frac{1}{1+t}$$

increases for large values of t since its derivative

$$g'(t) = \frac{t}{(1+t)^2}\left[\frac{3}{2\sqrt{\ln(1+t)}} - \frac{1}{2\ln^2(1+t)}\right] + \\ + \frac{1}{(1+t)^2}\left[\tfrac{3}{2}\sqrt{\ln(1+t)} - 1 + \frac{1}{2\ln(1+t)}\right]$$

is positive.

§ 6. *N*-functions which increase more rapidly than power functions

1. *The Δ_3-condition.* We shall say that the N-function $M(u)$ *satisfies the Δ_3-condition* if it is equivalent to the N-function $|u|M(u)$. Since $|u|M(u)$ is always $> M(u)$ for $u > 1$, the Δ_3-condition signifies that for values of u greater than some u_0, we have

$$|u|M(u) < M(ku), \tag{6.1}$$

where k is some constant.

If the N-function $M(u)$ satisfies the Δ_3-condition, then, as can easily be seen, all N-functions which are equivalent to $M(u)$ also satisfy this condition.

The N-functions $M(u)$ with principal parts e^u, e^{u^2}, $u^{\ln u}$, and so on, can serve as examples of N-functions which satisfy the Δ_3-condition inasmuch as condition (6.1) is manifestly satisfied for them.

The N-functions $M(u)$ increase more rapidly than any power function in all the examples introduced above. This is not accidental, inasmuch as every N-function $M(u)$ which satisfies the Δ_3-condition increases more rapidly than any power, u^n. In fact, in virtue of (6.1), for $u \geqslant u_0 k^n$, we have that

$$M(u) > \frac{u}{k}M\left(\frac{u}{k}\right) > \frac{u^2}{k^3}M\left(\frac{u}{k^2}\right) > \cdots > \\ > \frac{u^n}{k^{[n(n+1)]/2}}M\left(\frac{u}{k^n}\right) > \frac{M(u_0)}{k^{[n(n+1)]/2}}u^n.$$

However, not all N-functions which increase more rapidly than any power function satisfy the Δ_3-condition. For example, this condition is not satisfied by the N-function $M(u)$ for which p.p. $M(u) = u^{\sqrt{\ln u}}$ since for this N-function, we have that

$$\lim_{u\to\infty} \frac{M(ku)}{|u|M(u)} = 0$$

for arbitrary $k > 0$.

2. *Approximations for the complementary function.*

THEOREM 6.1. *Suppose the N-function $M(u)$ satisfies the Δ_3-condition. Then the N-function $N(v)$ complementary to it satisfies the inequalities*

$$k_1 v M^{-1}(k_1 v) \leqslant N(v) \leqslant k_2 v M^{-1}(k_2 v) \tag{6.2}$$

for large values of v, where $M^{-1}(v)$ is the function inverse to the function $M(u)$ and k_1, k_2 $(k_1 \leqslant k_2)$ are constants.

PROOF. We shall first show that the N-function

$$M_1(u) = \int_0^{|u|} M(t)\,dt$$

is equivalent to the N-function $M(u)$. In fact, by the condition of the theorem, for large u we have that

$$M_1(u) = \int_0^{u} M(t)\,dt < uM(u) \leqslant M(ku),$$

where k is a constant. On the other hand, for $u > 1$, we have

$$M_1(2u) = \int_0^{2u} M(t)\,dt > \int_u^{2u} M(t)\,dt > uM(u) > M(u).$$

Thus, $M_1(u) \sim M(u)$.

Therefore, the function $N_1(v)$ complementary to $M_1(u)$ is also equivalent to $N(v)$. The function $N_1(v)$ is found directly:

$$N_1(v) = \int_0^{|v|} M^{-1}(t)\,dt. \tag{6.3}$$

It follows from the last inequality that

$$N_1(v) < |v|M^{-1}(|v|)$$

and

$$N_1(v) = \int_0^{|v|} M^{-1}(t)\,dt > \int_{|v|/2}^{|v|} M^{-1}(t)\,dt > \frac{|v|}{2} M^{-1}\left(\frac{|v|}{2}\right).$$

Now (6.2) follows from the last inequalities and the fact that $N_1(v) \sim N(v)$. ★

We note that the left inequality in (6.2) is true for sufficiently large v for arbitrary function $M(u)$ (without the assumption about the Δ_3-condition) with arbitrary constant $k_1 < 1$. In fact, it follows from Young's inequality that for large v, we have

$$k_1 v M^{-1}(k_1 v) \leqslant k_1 N(v) + k_1^2 v < N(v).$$

3. *The construction of N-functions which are equivalent to the complementary functions.* As we have already noted, the construction of the complementary N-functions in the explicit form is possible only in rare cases. However, in applications, knowledge of exact formulas is not obligatory for the complementary function in many cases — it is sufficient to know the formula for some N-function which is equivalent to the one sought. It turns out that for certain classes of N-functions formulas for N-functions equivalent to the complementary functions can be shown. A class of this sort is isolated with the aid of the Δ_3-condition.

The next theorem follows directly from Theorem 6.1.

THEOREM 6.2. *Suppose the N-function $M(u)$ satisfies the Δ_3-condition and let the function $Q(v) = |v|M^{-1}(|v|)$ be the principal part of an N-function $\Psi(v)$. Then $\Psi(v) \sim N(v)$.*

In order that the function $Q(v)$ be the principal part of an N-function, it is sufficient that the function $Q'(v)$ increases monotonically to infinity as $v \to \infty$. Since $Q'(v) = M^{-1}(v) + v/p[M^{-1}(v)]$ for $v > 0$, we have that $\lim_{v\to\infty} Q'(v) = \infty$. In order that $Q'(v)$ increase monotonically, it is sufficient that $Q''(v)$ be a non-negative function for large values of v. The last condition is satisfied if

$$2p^2(u) - M(u)p'(u) \geqslant 0 \tag{6.4}$$

for large values of u.

This inequality is satisfied, for example, for the N-functions $M_1(u)$, $M_2(u)$, $M_3(u)$, for which p.p. $M_1(u) = e^u$, p.p. $M_2(u) = e^{u^2}$, and p.p. $M_3(u) = u^{\ln u}$. Therefore, the N-functions complementary to them are equivalent, respectively, to the N-functions $\Psi_1(v)$, $\Psi_2(v)$, $\Psi_3(v)$ for which

$$\text{p.p. } \Psi_1(v) = v \ln v, \quad \text{p.p. } \Psi_2(v) = v\sqrt{\ln v}, \qquad \text{p.p. } \Psi_3(v) = v^{1+(1/\sqrt{\ln v})}. \tag{6.5}$$

We showed, above, that N-functions $M(u)$ which satisfy the Δ_3-condition increase more rapidly than any power, $|u|^\alpha$ ($\alpha > 1$). This means that $M(u) > u^\alpha/\alpha$ ($u \geqslant u_0$), where u_0 is a non-negative number. Then the inequality $N(v) < v^\beta/\beta$ ($1/\alpha + 1/\beta = 1$) follows directly from Theorem 2.1. Thus, the N-function complementary to an N-function which satisfies the Δ_3-condition increases more slowly than any power, v^β ($\beta > 1$). For example, the functions (6.5) will be of this sort.

In considering arbitrary N-functions which increase more slowly than an arbitrary power function, the conjecture can be made that it is complementary to an N-function which satisfies the Δ_3-condition. This conjecture can be verified in a number of cases. We shall consider, for example, the N-function $N(v)$ for which the p.p. $N(v) = v(\ln v)^2$. We represent it in the form

$$\text{p.p. } N(v) = vQ^{-1}(v).$$

Then $Q(u) = e^{\sqrt{u}}$. Obviously, $Q(u)$ is the principal part of an N-function $M_1(u)$ which satisfies the Δ_3-condition. In virtue of Theorem 6.2, $N(v)$ is equivalent to $N_1(v)$.

The reasoning just carried out contains a general way of constructing N-functions which are equivalent to extensive classes of given N-functions, which increase more slowly than any power function. Suppose that such an N-function $N(v)$ is given. We represent it in the form $N(v) = |v|\,Q^{-1}(|v|)$. If the function $Q(u)$ turns out to be the principal part of the N-function $M_1(u)$ which satisfies the Δ_3-condition, then, in virtue of Theorem 6.2, $M_1(u) \sim M(u)$, where $M(u)$ is the complementary N-function to $N(v)$.

Another way of constructing an N-function equivalent to the complementary function to the N-function $N(v)$, which increases

more slowly than any power function v^β ($\beta > 1$) is also possible. Let $q(v) = N'(v)$, with the function $q^{-1}(u) = \text{p.p. } M_1(u)$, where $M_1(u)$ is an N-function satisfying the Δ_3-condition. Then, as was shown in the proof of Theorem 6.1, we have that

$$M_1(u) \sim M_2(u) = \int_0^{|u|} M_1(t)\, dt.$$

This means that the N-functions $N_1(v)$ and $N_2(v)$ which are complementary respectively to $M_1(u)$ and $M_2(u)$, where

$$N_2(v) = \int_0^{|v|} M_1^{-1}(s)\, ds,$$

are equivalent. Since for large values of v, we have that $M_1^{-1}(v) = q(v)$, then $N_2(v) \sim N(v)$ from which it follows that $M_2(u) \sim M(u)$. Consequently, the N-function $M_1(u)$ will be the function, equivalent to $M(u)$, sought.

4. *The composition of complementary functions.* Let $M(u)$ and $Q(u)$ be N-functions.

The N-functions $M(u)$ and $M[Q(u)]$ are never equivalent since, for arbitrary $k > 0$, we have that

$$\lim_{u\to\infty} \frac{M[Q(u)]}{M(ku)} = \infty.$$

This follows from the fact that, for sufficiently large u, we have $Q(u) > nku$, $M(nku) > nM(ku)$, where n is an arbitrary prescribed number, from which it follows that

$$\frac{M[Q(u)]}{M(ku)} > \frac{M(nku)}{M(ku)} > n.$$

The N-functions $M(u)$ and $Q[M(u)]$ can, however, be equivalent in certain cases. In this connection, if $M_1(u) \sim M(u)$ and $Q_1(u) \sim Q(u)$, then the functions $M_1(u)$ and $Q_1[M_1(u)]$ are equivalent if, and only if, the N-functions $M(u)$ and $Q[M(u)]$ are equivalent.

We make one further, obvious, observation: if $M(u) \sim Q[M(u)]$, then $M(u) \sim Q[Q[\cdots Q[M(u)]]]$.

THEOREM 6.3. *A necessary and sufficient condition that the N-function $M(u)$ satisfy the Δ_3-condition is that the relation $N[M(u)] \sim M(u)$ hold, where $N(v)$ is the function complementary to $M(u)$.*

PROOF. Suppose $M(u)$ satisfies the Δ_3-condition. We shall prove that $N_1[M(u)] \sim M(u)$, where $N_1(v)$ is the N-function defined by equality (6.3). Since $N_1(v) \sim N(v)$, this will prove the necessity of the condition of the theorem.

By the definition of the function $N_1(v)$, we have that

$$N_1(v) \leqslant |v|M^{-1}(|v|),$$

from which it follows that $N_1[M(u)] \leqslant M(u)u \leqslant M(ku)$, for large values of u, in virtue of the fact that $|u|M(u) \sim M(u)$.

On the other hand, for arbitrary N-function $N_1(v)$, we have $N_1[M(u)] > M(u)$ for large values of the argument.

Thus, $N_1[M(u)] > M(u)$.

We now prove the sufficiency of the condition of the theorem. Suppose $N[M(u)] \sim M(u)$. This signifies that, for large values of the argument, we have $N[M(u) \leqslant M(k_1u)$. Since, in virtue of Young's inequality, $vM^{-1}(v) \leqslant N(v) + v < 2N(v)$ for large values of the argument, we have that

$$uM(u) \leqslant 2N[M(u)] \leqslant 2M(k_1u) < M(2k_1u)$$

for large values of u.

Thus, the N-function $M(u)$ satisfies the Δ_3-condition. ★

Suppose $M(u)$ and $Q(u)$ are two N-functions, the first of which satisfies the Δ_3-condition. We shall show that both compositions $M[Q(u)]$ and $Q[M(u)]$ also satisfy the Δ_3-condition. This follows from the following chain of obvious inequalities which are valid for large values of the argument:

$$uM[Q(u)] \leqslant Q(u)M[Q(u)] \leqslant M[kQ(u)] \leqslant M[Q(ku)]$$

and

$$uQ[M(u)] \leqslant Q[uM(u)] \leqslant Q[M(ku)].$$

5. *The Δ^2-condition.* In a number of cases, the functions $M(u)$ and $Q[M(u)]$ are also equivalent if, and only if, $Q(v)$ is a function which increases "essentially" more rapidly than the complementary function $N(v)$ to $M(u)$. In the sequel, we shall be interested in the case when $Q(v) = v^2$.

We shall say that the N-function $M(u)$ *satisfies the* Δ^2*-condition* if $M(u) \sim M^2(u)$, i.e. if there exists a $k > 1$ such that

$$M^2(u) \leqslant M(ku) \tag{6.6}$$

for all sufficiently large values of u.

It is easily seen that the N-function $M(u)$ satisfies the Δ^2-condition if $M(u) \sim M^\alpha(u)$ for some $\alpha > 1$. Conversely, if the N-function $M(u)$ satisfies the Δ^2-condition, then $M(u) > M^\alpha(u)$ for arbitrary $\alpha > 1$.

It can be verified directly that the N-function $M(u)$ satisfies the Δ^2-condition if this condition is satisfied by any N-function which is equivalent to $M(u)$.

The N-functions with principal parts e^u, e^{u^2}, and so on, can serve as examples of functions which satisfy the Δ^2-condition.

If the N*-function* $M(u)$ *satisfies the* Δ^2*-condition, then it also satisfies the* Δ_3*-condition.* In fact, the Δ^2-condition implies the existence of a $k > 1$ such that $M(ku) \geqslant M^2(u)$ for large values of u, and since $M(u) > u$ for large values of u we have that $M(ku) > uM(u)$, i.e. inequality (6.1) is satisfied. But this means that the N-function $M(u)$ satisfies the Δ_3-condition.

However, the class of N-functions, satisfying the Δ_3-condition, is more extensive than the class of N-functions satisfying the Δ^2-condition. For example, the N-function $M(u)$ with the principal part $u^{\ln u}$ satisfies the Δ_3-condition but it does not satisfy the Δ^2-condition since, for arbitrary $k > 0$, we have that

$$\lim_{u \to \infty} \frac{u^{2 \ln u}}{(ku)^{\ln (ku)}} = \infty.$$

We showed, above, that every N-function which satisfies the Δ_2-condition increases more slowly than any power function. This fact is not only true for N-functions. In fact, let $f(x)$ be an arbitrary non-negative non-decreasing function which satisfies the inequality $f(2u) \leqslant kf(u)$ for $u \geqslant u_0$. Then it follows from $2^n u_0 < u \leqslant 2^{n+1} u_0$ that $k^n < (u/u_0)^{\ln_2 k}$ and

$$f(u) \leqslant k^{n+1} f(u_0) \leqslant kf(u_0)(u/u_0)^{\ln_2 k}.$$

We shall now consider a non-decreasing function $f(u)$ which satisfies the inequality

$$2f(u) < f(ku) \tag{6.7}$$

for $u \geqslant u_0$. Clearly, $k > 1$. Suppose $k^n u_0 < u \leqslant k^{n+1} u_0$. Then

$$2^n \geqslant \{u/(ku_0)\}^{\ln_k 2}$$

and

$$f(u) > f(k^n u_0) > 2^n f(u_0) \geqslant f(u_0)\{u/(ku_0)\}^{\ln_k 2}.$$

Thus, (6.7) implies that

$$f(u) > u^\alpha \tag{6.8}$$

for large values of u, where $\alpha < \ln_k 2$.

LEMMA 6.1. *Suppose the positive non-decreasing function $p(u)$ is greater than unity and satisfies the inequality*

$$p^2(u) < p(ku) \tag{6.9}$$

for large values of the argument. Then there exists an $\alpha > 0$ such that $p(u) > e^{u^\alpha}$ for large values of u.

PROOF. It follows from the condition of the lemma that the function $f(u) = \ln p(u)$ satisfies inequality (6.7). Therefore, it satisfies inequality (6.8), i.e. $\ln p(u) > u^\alpha$. ★

It follows from this lemma that every N-function, which satisfies the Δ^2-condition, increases more rapidly than some function e^{u^α} for large values of the argument.

THEOREM 6.4. *Suppose the right derivative $p(u)$ of the N-function $M(u)$ satisfies condition (6.9) for $u \geqslant u_0$. Then $M(u)$ satisfies the Δ^2-condition.*

PROOF. In virtue of Lemma 6.1, we may assume that $2u < p(ku)$ for $u \geqslant u_0$. Therefore, for $u \geqslant u_0$, we have

$$2up^2(u) < 2up(ku) < p^2(ku) < p(k^2 u).$$

From this and the inequality $M(u) < up(u)$, it follows that

$$M^2(u) = 2\int_0^u M(t)p(t)\,dt \leqslant M^2(u_0) + 2\int_{u_0}^u tp^2(t)\,dt <$$

$$< M^2(u_0) + \int_0^u p(k^2 t)\,dt < M^2(u_0) + M(k^2 u)$$

for $u > u_0$.

For large values of u, $M^2(u_0) < M(k^2 u)$. It therefore follows from the preceding inequality that $M^2(u) < 2M(k^2 u) < M(2k^2 u)$. ★

As we have already remarked, an N-function which satisfies the Δ^2-condition increases more rapidly than some function of the form e^{u^α} $(\alpha > 0)$. It turns out that the converse does not hold. We leave it to the reader to construct an example of an N-function which increases more rapidly than some function e^{u^α} $(\alpha > 0)$ and satisfies neither the Δ^2-condition nor the Δ_3-condition.

The following assertion yields a simple criterion that the N-function $M(u)$ satisfies the Δ^2-condition.

Suppose an $\alpha > 0$ can be found such that the function

$$\varphi(u) = \frac{\ln M(u)}{u^\alpha} \tag{6.10}$$

does not decrease for values of u larger than some u_0. Then the N-function $M(u)$ satisfies the Δ^2-condition.

In fact, suppose $u \geqslant u_0$. Then

$$M^2(u) = e^{2\ln M(u)} = e^{2u^\alpha[\ln M(u)/u^\alpha]} \leqslant e^{2u^\alpha[\ln M(2^{1/\alpha}u)/2u^\alpha]} = M(2^{1/\alpha}u).$$

Since, moreover, for sufficiently large values of u, $M(u) < M^2(u)$, we have that $M(u) \sim M^2(u)$, which was to be proved.

As we noted, both compositions $M[Q(u)]$ and $Q[M(u)]$ of the N-functions $M(u)$ and $Q(u)$ satisfy the Δ_3-condition if the N-function $M(u)$ satisfies this condition. We shall now assume that $M(u)$ satisfies the Δ^2-condition. Then $M[Q(u)]$ also satisfies the Δ^2-condition inasmuch as for large values of u we have

$$M^2[Q(u)] < M[kQ(u)] < M[Q(ku)].$$

It turns out that the N-function $Q[M(u)]$, in this connection, might not satisfy the Δ^2-condition. Moreover, whatever the function $M(u)$, we can always construct a function $Q(u)$ such that $Q[M(u)]$ does not satisfy the Δ^2-condition. Suppose $0 < v_0 < M(v_0) < v_1 < M(v_1) < \cdots < v_n < M(v_n) < \cdots$. We define the N-function $Q(u)$ by setting it equal to u^2 for $0 < u < M(v_0)$ and defining it as the linear function $Q(v_{n+1}) + k_n[u - M(v_{n-1})]$ on every closed interval $M(v_{n-1}) \leqslant u \leqslant M(v_n)$. The angular coefficients k_n are chosen so that they increase — this guarantees the convexity of the function $Q(u)$. Furthermore, we shall require that these angular coefficients increase so rapidly that the inequality

$$\{Q(v_{n-1}) + k_n[v_n - M(v_{n-1})]\}^2 > Q(v_{n-1}) + k_n[M(v_n) - M(v_{n-1})]$$

is satisfied for all n. Then the N-function $Q(u)$ will satisfy the inequalitites $Q^2(v_n) > Q[M(v_n)]$ $(n = 1, 2, \cdots)$.

We set $v_n = M(u_n)$. Without loss of generality, it can be assumed that $M(u_n) > nu_n$. Then it follows from the inequalities obtained that $Q^2[M(u_n)] > Q\{M[M(u_n)]\} > Q[M(nu_n)]$.

This means that the composition $Q[M(u)]$ does not satisfy the Δ^2-condition.

6. *Properties of the complementary functions.*

THEOREM 6.5. *Suppose the N-function $N(v)$ satisfies the Δ_3-condition. Then the N-function $N(v)$, complementary to $M(u)$, satisfies the Δ_2-condition.*

PROOF. Let k_1 and k_2 be the constants appearing in (6.2). Since the function $M^{-1}(v)$ is convex and $(2k_2/k_1) > 1$, we have that

$$M^{-1}\left(\frac{2k_2}{k_1}v\right) < \frac{2k_2}{k_1}M^{-1}(v).$$

Consequently, in virtue of (6.2), we have that

$$N(2v) \leqslant 2k_2vM^{-1}(2k_2v) < 2k_2v\frac{2k_2}{k_1}M^{-1}(k_1v) \leqslant \left(\frac{2k_2}{k_1}\right)^2 N(v)$$

for large values of v, from which the assertion of the theorem does follow.

THEOREM 6.6. *Suppose the N-function $M(u)$ satisfies the Δ^2-condition. Then the N-function $N(v)$ complementary to it satisfies the Δ'-condition.*

PROOF. Since $M(u)$ satisfies the Δ^2-condition, constants t_0, k can be found such that $M(kt) \geqslant M^2(t)$ for $t \geqslant t_0$. Without loss of generality, we may assume that $t_0 > 1$.

Let $t \geqslant s \geqslant t_0$. Then $M(kts) > M(kt) > M^2(t) \geqslant M(t)M(s)$. Setting $t = M^{-1}(u)$, $s = M^{-1}(v)$ in the last inequality, we obtain that $M^{-1}(uv) \leqslant kM^{-1}(u)M^{-1}(v)$ for $u, v \geqslant M(t_0)$.

Now u and v occur symmetrically in the last inequality — therefore it is valid for all $u, v \geqslant M(t_0)$.

Since the N-function $M(u)$ satisfies the Δ^2-condition, it also satisfies the Δ_3-condition. In virtue of (6.2), a $u_0 \geqslant M(t_0) + 1$ can be found such that $k_1uM^{-1}(k_1u) \leqslant N(u) \leqslant k_2uM^{-1}(k_2u)$ for $u \geqslant u_0$ (or $v \geqslant u_0$), where k_1 and k_2 are certain constants. Con-

sequently, for $u, v \geqslant u_0$, we have

$$N(uv) \leqslant k_2 uvM^{-1}(k_2 uv) = (\sqrt{k_2}u)(\sqrt{k_2}v)M^{-1}(\sqrt{k_2}u\sqrt{k_2}v),$$

from which it follows that

$$N(uv) \leqslant k\sqrt{k_2}uM^{-1}(\sqrt{k_2}u)\sqrt{k_2}vM^{-1}(\sqrt{k_2}v)$$

and, finally, that

$$N(uv) \leqslant kN\{\sqrt{k_2}/(k_1 u)\}N\{\sqrt{k_2}/(k_1 v)\}.$$

In virtue of the preceding theorem, the N-function $N(v)$ satisfies the Δ_2-condition. It therefore follows from the last inequality that $N(uv) \leqslant cN(u)N(v)$ for large values of u, v, where c is a certain constant. ★

THEOREM 6.7. *Suppose the N-function $M(u)$ satisfies the Δ_3-condition. A necessary and sufficient condition that the complementary N-function $N(v)$ to $M(u)$ satisfy the Δ'-condition is that the inequality*

$$(Muv) \geqslant M(\alpha u)M(\beta v) \tag{6.11}$$

be satisfied for large values of u and v, where α, β are certain constants.

PROOF. Suppose condition (6.11) is satisfied. Then the inequality

$$M^{-1}(uv) \leqslant \frac{M^{-1}(u)M^{-1}(v)}{\alpha\beta}$$

holds for large u, v. From this inequality and Theorem 6.2 it follows that

$$N(uv) \leqslant k_2 uvM^{-1}(k_2 uv) \leqslant \frac{k_2 uv}{\alpha\beta} M^{-1}(k_2 u)M^{-1}(v) \leqslant$$

$$\leqslant \frac{1}{\alpha\beta} N\left(\frac{k_2}{k_1}u\right)N\left(\frac{1}{k_1}v\right)$$

and since, in virtue of Theorem 6.5, $N(v)$ satisfies the Δ_2-condition, we have that $N(uv) \leqslant cN(u)N(v)$.

This proves the sufficiency of condition (6.11).

Let us assume that $N(v)$ satisfies the Δ'-condition. This signifies, in virtue of Theorem 6.2, that $uvM^{-1}(uv) \leqslant kuM^{-1}(u)vM^{-1}(v)$ for large values of u, v. It follows from this that $M(kuv) \geqslant M(u)M(v)$ for large values of the argument. ★

This theorem and Theorem 6.6 imply that inequality (6.11) is satisfied for N-functions which satisfy the Δ^2-condition. A much stronger assertion holds: if $M(u)$ satisfies the Δ^2-condition, then a $u_0 > 0$ can be found such that

$$M(uv) \geqslant M(u)M(v) \tag{6.12}$$

for $u, v \geqslant u_0$.

As u_0, we must take a number such that $M^2(u) \leqslant M(u_0 u)$ for $u \geqslant u_0$. Then, for $u \geqslant v \geqslant u_0$, we have that

$$M(u)M(v) \leqslant M^2(u) \leqslant M(u_0 u) \leqslant M(uv).$$

Condition (6.11) is usually easily verified. Let us consider, for example, the N-function $M(u)$ for which p.p. $M(u) = u^{\ln u}$. It satisfies the Δ_2-condition. For it, condition (6.11) signifies that

$$e^{(\ln u + \ln v)^2} > e^{\ln^2 u}\, e^{\ln^2 v}$$

for large u, v.

7. *Test for the Δ^2-condition for the complementary function.* In a number of cases. the occasion arises to study N-functions $M(u)$ for which an explicit expression is not known, but a formula for the complementary function $N(v)$ is given. The question arises (which was solved earlier in the study of other classes of N-functions) of how to determine from the function $N(v)$ whether or not the complementary function $M(u)$ satisfies the Δ^2-condition.

We first establish a lemma concerning arbitrary N-functions.

Suppose $\Phi(u)$ is an N-function. In virtue of (1.18), the function $\Phi(u)/u$ increases monotonically for $u > 0$, where $\lim\limits_{u \to 0} \{\Phi(u)/u\} = 0$ and $\lim\limits_{u \to \infty} \{\Phi(u)/u\} = \infty$. Therefore, the function

$$\Phi_1(u) = \int_0^{|u|} \frac{\Phi(t)}{t}\, dt$$

is an N-function.

LEMMA 6.2. $\Phi_1(u) \sim \Phi(u)$.

PROOF. Clearly, $\Phi_1(u) \leqslant \Phi(u)$. On the other hand, for $u > 0$, we have that

$$\Phi_1(u) = \int_0^u \frac{\Phi(t)}{t}\, dt > \int_{u/2}^u \frac{\Phi(t)}{t}\, dt > \Phi\left(\frac{u}{2}\right). \quad \bigstar$$

THEOREM 6.8. *A necessary and sufficient condition that the N-function $M(u)$ satisfy the Δ^2-condition is that the complementary N-function $N(v)$ to it satisfy the inequality*

$$\frac{N(v)}{v} < k \frac{N(\sqrt{v})}{\sqrt{v}} \tag{6.13}$$

for large values of the argument, where k is some constant.

Proof of necessity. Satisfaction of the Δ^2-condition means that $M^2(u) < M(k_1 u)$ for large values of u. From this it follows that, for large values of the argument, we have

$$M^{-1}(v) < k_1 M^{-1}(\sqrt{v}), \tag{6.14}$$

where $M^{-1}(v)$ is the function inverse to $M(u)$.

Since the function $M(u)$ satisfies the Δ_3-condition, we have, in virtue of Theorem 6.1, for large values of the argument, that

$$vM^{-1}(v) < N(k_2 v) \tag{6.15}$$

and that

$$N(v) < k_3 v M^{-1}(k_3 v). \tag{6.16}$$

In virtue of (6.16) and (6.14), we have that

$$\frac{N(v)}{v} < k_3 M^{-1}(k_3 v) < k_1 k_2 M^{-1}(\sqrt{k_3 v})$$

and, in virtue of (6.15), that

$$\frac{N(v)}{v} < \frac{k_1 k_3}{\sqrt{k_3 v}} N(k_2 \sqrt{k_3 v}). \tag{6.17}$$

In virtue of Theorem 6.5, the function $N(v)$ satisfies the Δ_2-condition, i.e. $N(k_2 \sqrt{k_3} \sqrt{v}) < k_4 N(\sqrt{v})$ for large values of the argument. Therefore, (6.17) implies (6.13), where $k = k_1 k_4 \sqrt{k_3}$.

Proof of sufficiency. We consider the function $r(v) = N(v)/v$. In virtue of Lemma 6.2, the N-function

$$N_1(v) = \int_0^{|v|} r(t)\, dt$$

is equivalent to the function $N(v)$. The N-function $M_1(u)$ which

is complementary to $N_1(v)$ is computed directly:

$$M_1(u) = \int_0^{|u|} r^{-1}(t)\,dt,$$

where $r^{-1}(t)$ is the function which is the inverse to the monotonically increasing function $r(t)$. The N-function $M_1(u)$ is equivalent to the N-function $M(u)$.

In virtue of (6.13), for large values of the argument, we have that $[r^{-1}(u)]^2 < r^{-1}(ku)$. From this inequality and Theorem 6.4, it follows that the N-function $M_1(u)$ satisfies the Δ^2-condition. This signifies that $M(u)$ also satisfies this condition. ★

We note that k is always > 1 in condition (6.13) inasmuch as for an arbitrary N-function, for $v > 1$, we have that

$$\frac{N(v)}{v} > \frac{N(\sqrt{v})}{\sqrt{v}}.$$

Functions, which are complementary to those which satisfy the Δ^2-condition, constitute a part of the class of N-functions which increase more slowly than an arbitrary power function $|v|^\alpha$ $(\alpha > 1)$. Furthermore, it follows from Lemma 6.1, Theorem 2.1 and Theorem 6.1 that such N-functions satisfy the inequality $N(v) < v \ln^\beta v$ $(\beta > 0)$ for large values of the argument.

8. *Further discussion on the composition of N-functions.* In the present subsection we shall establish some properties of the composition $N_1[N_2(v)]$ of the N-functions $N_1(v)$ and $N_2(v)$ which are complementary to the N-functions $M_1(u)$ and $M_2(u)$ which satisfy the Δ_3-condition.

We shall first prove an assertion which is valid for arbitrary N-functions.

LEMMA 6.3. *Suppose $\Phi_1(v)$ and $\Phi_2(v)$ are two N-functions. Then the function $\Phi(v) = \Phi_1(v)\Phi_2(v)/|v|$ is also an N-function.*

PROOF. The function $\Phi(v)$ is even, non-negative and manifestly satisfies the conditions

$$\lim_{v\to 0} \frac{\Phi(v)}{v} = 0, \quad \lim_{v\to\infty} \frac{\Phi(v)}{v} = \infty.$$

In virtue of the second definition of an N-function (see p. 9),

it suffices to prove that $\Phi(v)$ is a convex function, i.e. that

$$\Phi\left(\frac{v_1 + v_2}{2}\right) \leqslant \tfrac{1}{2}[\Phi(v_1) + \Phi(v_2)]. \tag{6.18}$$

In virtue of (1.18), the functions $\Phi_1(v)/v$ and $\Phi_2(v)/v$ increase monotonically for positive values of v. Therefore, the function $\Phi(v)$ also increases monotonically. Consequently, it suffices to prove inequality (6.18) for positive v_1 and v_2.

Clearly,

$$\left[\frac{\Phi_1(v_1)}{v_1} - \frac{\Phi_1(v_2)}{v_2}\right]\left[\frac{\Phi_2(v_1)}{v_1} - \frac{\Phi_2(v_2)}{v_2}\right] \geqslant 0,$$

inasmuch as both factors have the same sign. From this it follows that

$$\frac{[\Phi_1(v_1) + \Phi_1(v_2)][\Phi_2(v_1) + \Phi_2(v_2)]}{v_1 + v_2} \leqslant \frac{\Phi_1(v_1)\Phi_2(v_1)}{v_1} + \frac{\Phi_1(v_2)\Phi_2(v_2)}{v_2}$$

and, in virtue of the convexity of the N-functions $\Phi_1(v)$ and $\Phi_2(v)$,

$$\Phi\left(\frac{v_1 + v_2}{2}\right) = \frac{2}{v_1 + v_2}\Phi_1\left(\frac{v_1 + v_2}{2}\right)\Phi_2\left(\frac{v_1 + v_2}{2}\right) \leqslant$$

$$\leqslant \frac{1}{2(v_1+v_2)}[\Phi_1(v_1) + \Phi_1(v_2)][\Phi_2(v_1) + \Phi_2(v_2)] \leqslant$$

$$\leqslant \frac{1}{2}\left[\frac{\Phi_1(v_1)\Phi_2(v_1)}{v_1} + \frac{\Phi_1(v_2)\Phi_2(v_2)}{v_2}\right],$$

from which (6.18) does indeed follow. ★

THEOREM 6.9. *Suppose the N-functions $M_1(u)$ and $M_2(u)$ satisfy the Δ_3-condition and that $M_1(u) \prec M_2(u)$. Then*

$$N_1[N_2(v)] \sim \Phi(v) = \frac{N_1(v)N_2(v)}{|v|}.$$

PROOF. We first note that the relation

$$\frac{N_1(v)N_2(v)}{|v|} \prec N_1[N_2(v)]$$

is valid for arbitrary N-functions $N_1(v)$ and $N_2(v)$. In fact, in

virtue of (1.16), the inequality $N_1(v)/v < N_1[N_2(v)]/N_2(v)$ is valid for $N_2(v) > v$, from which it follows that $N_1(v)N_2(v)/v < N_1[N_2(v)]$ for these values of v.

We shall prove that, under the conditions of the theorem, we have $N_1[N_2(v)] \prec N_1(v)N_2(v)/|v|$.

The relation $M_1(u) \prec M_2(u)$ signifies that $M_1(u) \leqslant M_2(k_1u)$ for large values of the argument, where k_1 is a number which can be assumed larger than unity. Therefore, $M_2^{-1}(v) \leqslant k_1M_1^{-1}(v)$ for large values of the argument. From this it follows that

$$vM_2^{-1}(v) \leqslant k_1M_1^{-1}(v)M_1[M_1^{-1}(v)]. \qquad (*)$$

The N-function $M_1(u)$ satisfies the Δ_3-condition. Therefore, a $k_2 > 1$ can be found such that $uM_1(u) \leqslant M_1(k_2u)$ for large values of the argument. It then follows from the preceding inequality (*) that

$$vM_2^{-1}(v) \leqslant k_1M_1[k_2M_1^{-1}(v)] \leqslant M_1[k_1k_2M_1^{-1}(v)] \qquad (6.19)$$

for large values of the argument. Consequently,

$$M_1^{-1}[vM_2^{-1}(v)] \leqslant k_1k_2M_1^{-1}(v)$$

and

$$vM_2^{-1}(v)M_1^{-1}[vM_2^{-1}(v)] \leqslant k_1k_2\frac{vM_1^{-1}(v)vM_2^{-1}(v)}{v}. \qquad (6.20)$$

In virtue of Theorem 6.1, constants $k_3 < 1$ and $k_4 > 1$ can be found such that

$$N_1(k_3v) \leqslant vM_1^{-1}(v) \leqslant N_1(k_4v),$$
$$N_2(k_3v) \leqslant vM_2^{-1}(v) \leqslant N_2(k_4v)$$

for large values of the argument.

From these inequalities and (6.20), it follows that

$$N_1[k_3N_2(k_3v)] \leqslant k_1k_2\frac{N_1(k_4v)N_2(k_4v)}{v} = k_1k_2k_4\Phi(k_4v)$$

for large values of the argument. It follows from this that

$$N_1[N_2(k_3^2v)] \leqslant \Phi(k_1k_2k_4^2v).$$

Thus, $N_1[N_2(v)] \prec \Phi(v)$. ★

THEOREM 6.10. *Suppose $M_1(u)$ satisfies the Δ^2-condition and that $M_2(u)$ satisfies the Δ_3-condition. Then the assertion of Theorem* 6.9: $N_1[N_2(v)] \sim \Phi(v) = N_1(v)N_2(v)/|v|$ *is valid.*

PROOF. Since $M_2(u) > u$ for large values of the argument, we have that $vM_2^{-1}(v) < v^2 = M_1^2[M_1^{-1}(v)]$ for large v.

The N-function $M_1(u)$ satisfies the Δ^2-condition. This means that there exists a $k_1 > 1$ such that $M_1^2(u) \leqslant M_1(k_1u)$ for large values of the argument. Therefore $vM_2^{-1} \leqslant M_1[k_1M_1^{-1}(v)]$ for large values of v. This inequality coincides with inequality (6.19). From it, as was also the case in the proof of the preceding theorem, it follows that $N_1[N_2(v)] \prec \Phi(v)$. The relation $\Phi(v) \prec N_1[N_2(v)]$, as was pointed out above, is always valid. ★

The next theorem follows from Theorem 6.10.

THEOREM 6.11. *If the N-functions $M_1(u)$ and $M_2(u)$ satisfy the Δ^2-condition, then*

$$N_1[N_2(v)] \sim N_2[N_1(v)] \sim \Phi(v) = \frac{N_1(v)N_2(v)}{|v|}.$$

It is not known whether it is sufficient in Theorem 6.11 to require the satisfaction of a weaker Δ_3-condition.

As an example, we consider N-functions $N_1(v)$ and $N_2(v)$ such that

$$\text{p.p. } N_1(v) = v \ln v, \quad \text{p.p. } N_2(v) = ve^{\sqrt{\ln v}}.$$

The function $M_1(u)$ satisfies the Δ^2-condition; $M_2(u)$ satisfies the Δ_3-condition but it does not satisfy the Δ^2-condition. In this connection, $M_2(u) \prec M_1(u)$ inasmuch as $N_1(v) \prec N_2(v)$. In virtue of Theorem 6.9, we have that $N_2[N_1(v)] \sim N_1(v)N_2(v)/|v|$ and, in virtue of Theorem 6.10, that $N_1[N_2(v)] \sim N_1(v)N_2(v)/|v|$.

Thus, the assertion of Theorem 6.11 is true in this example although its conditions are not satisfied.

Theorem 6.11 is extended in a natural way by the following assertion:

THEOREM 6.12. *Suppose the N-functions $M_1(u)$ and $M_2(u)$ satisfy the Δ^2-condition. Then the N-functions complementary to the N-functions $N_1[N_2(v)]$ and $N_2[N_1(v)]$ also satisfy the Δ^2-condition.*

PROOF. In virtue of Theorem 6.11, it suffices to consider the N-function $\Psi(u)$ which is complementary to the N-function

$$\Phi(v) = \frac{N_1(v)N_2(v)}{|v|}.$$

In virtue of Theorem 6.8,

$$\frac{N_1(v)}{v} < k_1 \frac{N_1(\sqrt{v})}{\sqrt{v}}, \qquad \frac{N_2(v)}{v} < k_2 \frac{N_2(\sqrt{v})}{\sqrt{v}}$$

for large values of the argument. Therefore,

$$\frac{\Phi(v)}{v} = \frac{N_1(v)}{v} \frac{N_2(v)}{v} < k_1 k_2 \frac{N_1(\sqrt{v})}{\sqrt{v}} \frac{N_2(\sqrt{v})}{\sqrt{v}} = k_1 k_2 \frac{\Phi(\sqrt{v})}{\sqrt{v}},$$

from which it follows, in virtue of Theorem 6.8 again, that $\Psi(u)$ satisfies the Δ^2-condition. ★

§ 7. Concerning a class of N-functions

1. *Formulation of the problem.* In the preceding section, we pointed out formulas for N-functions which are equivalent to the complementary functions to certain N-functions $M(u)$. In this connection, we succeeded in considering only those N-functions which either increase more rapidly than an arbitrary exponential function or more slowly than all exponential functions of the form $u^{1+\varepsilon}$ ($\varepsilon > 0$). To such N-functions there do not correspond functions $M(u)$ for which

$$\text{p.p. } M(u) = \frac{u^\alpha}{\alpha} (\ln u)^{\gamma_1} (\ln \ln u)^{\gamma_2} \cdots (\ln \ln \cdots \ln u)^{\gamma_n}, \quad (7.1)$$

where $\alpha > 1, \gamma_1, \gamma_2, \cdots, \gamma_n$ are arbitrary numbers.

In the present section we study the special class of N-functions containing functions with principal parts of the form (7.1). For functions in this class, one succeeds in effectively constructing N-functions equivalent to the complementary functions.

For simplicity of discussion we assume that all N-functions considered in the present section have ordinary (and not only right) derivatives for large values of the argument.

2. *The class* $\mathfrak{M}$. Below, by $\kappa_R(u)$ we denote the function

$$\kappa_R(u) = \frac{ur(u)}{R(u)}, \quad (7.2)$$

where $R(u)$ is a differentiable function and $r(u)$ is its derivative.

It is clear that the function $\varkappa_R(u)$ is defined for those values of u for which $r(u)$ exists and for which $R(u) \neq 0$.

The following simplest properties of the function $\varkappa_R(u)$ are evident:

$$\varkappa_{R_1 \cdot R_2}(u) = \varkappa_{R_1}(u) + \varkappa_{R_2}(u), \tag{7.3}$$

$$\varkappa_{R_1[R_2]}(u) = \varkappa_{R_1}[R_2(u)] \cdot \varkappa_{R_2}(u). \tag{7.4}$$

Both these formulas are valid for those values of u for which the expressions in the right members have meaning.

We note that, for arbitrary differentiable N-function $M(u)$,

$$\varkappa_M(u) > 1. \tag{7.5}$$

In fact, since $p(u) = M'(u)$ increases, we have that

$$up(u) > M(u) = \int_0^{|u|} p(t)\,dt,$$

from which (7.5) does indeed follow.

We denote by $\mathfrak{M}$ the class of those functions $R(u)$ for which $\varkappa_R(u)$ is defined for all large u and

$$\lim_{u \to \infty} \varkappa_R(u) = 0. \tag{7.6}$$

In virtue of (7.3), whenever the class $\mathfrak{M}$ contains the two functions $R_1(u)$ and $R_2(u)$ it also contains their product $R_1(u)R_2(u)$. It follows from this same property (7.3) that $\varkappa_{1/R}(u) = -\varkappa_R(u)$, from which it in turn follows that the class $\mathfrak{M}$ contains the function $1/R(u)$ whenever it contains the function $R(u)$.

In virtue of (7.4), the superposition $R_1[R_2(u)]$ belongs to $\mathfrak{M}$ if $R_2(u) \in \mathfrak{M}$, $\lim_{u \to \infty} R_2(u) = \infty$, and $\overline{\lim_{u \to \infty}}\, \varkappa_{R_1}(u) < \infty$.

It follows from the properties of the class $\mathfrak{M}$ described above that the functions $(\ln u)^{\gamma_1}$, $(\ln \ln u)^{\gamma_2}$, $\cdots$, $(\ln \ln \cdots \ln u)^{\gamma_n}$ belong to this class (where the γ_i are arbitrary numbers).

Suppose $\varepsilon > 0$ is given. Then for the function $R(u) \in \mathfrak{M}$ one can find a u_0 such that $|ur(u)/R(u)| < \varepsilon$ $(u \geqslant u_0)$, from which it follows that $r(u)/R(u) < \varepsilon/u$ $(u \geqslant u_0)$. Integrating the last inequality from u_0 to u, we obtain

$$\ln \left| \frac{R(u)}{R(u_0)} \right| < \varepsilon \ln \frac{u}{u_0}.$$

Consequently,

$$|R(u)| < |R(u_0)|\left(\frac{u}{u_0}\right)^{\varepsilon} \quad (u \geqslant u_0). \tag{7.7}$$

The relation

$$\lim_{u\to\infty} \frac{u}{|R(u)|} = \infty, \tag{7.8}$$

which will be of importance in the sequel, follows from (7.7).

LEMMA 7.1. *Let the function $R(u) \in \mathfrak{M}$ be positive for large values of u. Then, for arbitrary $\varepsilon > 0$, the function $u^{\varepsilon}R(u)$ increases monotonically to infinity.*

PROOF. It is sufficient to show that the function $h(u) = u^{\varepsilon/2}R(u)$ has a positive derivative for large values of u. But this follows from (7.6) since

$$h'(u) = \frac{\varepsilon}{2}u^{(\varepsilon/2)-1}R(u) + u^{\varepsilon/2}r(u) = u^{(\varepsilon/2)-1}R(u)\left[\frac{\varepsilon}{2} + \kappa_R(u)\right]. \bigstar$$

LEMMA 7.2. *Suppose $R(u)$ is a function from the class $\mathfrak{M}$ such that the functions*

$$\frac{u^{\alpha}}{\alpha}R(u), \quad \frac{v^{\beta}}{\beta R^{\beta-1}(v)} \quad \left(\alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1\right)$$

are the principal parts of the N-functions $M(u)$ and $N_1(v)$, respectively. Assume that the condition

$$\lim_{v\to\infty} \frac{R\left[\dfrac{v^{\beta-1}}{R^{\beta-1}(v)}\right]}{R(v)} = b > 0 \tag{7.9}$$

is satisfied. Then the N-function $N_1(v)$ is equivalent to the N-function $N(v)$ which is complementary to the N-function $M(u)$.

PROOF. We consider the functions

$$p_2(u) = u^{\alpha-1}R(u), \quad q_3(v) = \left[\frac{v}{R(v)}\right]^{\beta-1}.$$

These functions increase monotonically to infinity for large values of u, v, in virtue of Lemma 7.1. Therefore, each of them can be considered as the principal part of the derivatives of certain N-functions $M_2(u)$ and $N_3(v)$.

A direct calculation shows that

$$\frac{p(u)}{p_2(u)} = 1 + \frac{1}{\alpha}\kappa_R(u), \quad \frac{q_1(v)}{q_3(v)} = 1 - \frac{\beta - 1}{\beta}\kappa_R(u)$$

for large u, where $p(u) = M'(u)$, $q_1(v) = N_1'(v)$. From the last equalities and (7.6), it follows that

$$\lim_{u\to\infty} \frac{p(u)}{p_2(u)} = \lim_{v\to\infty} \frac{q_1(v)}{q_3(v)} = 1.$$

Therefore, in virtue of Lemma 3.2, we have

$$M(u) \sim M_2(u), \quad N_1(v) \sim N_3(v). \tag{7.10}$$

Since

$$p_2[q_3(v)] = \frac{vR\left\{\left[\frac{v}{R(v)}\right]^{\beta-1}\right\}}{R(v)},$$

we have, in virtue of (7.9), that

$$\lim_{v\to\infty} \frac{p_2[q_3(v)]}{v} = b > 0,$$

and it follows from Theorems 3.4 and 3.2 that

$$M_2(u) \sim M_3(u), \quad N_2(v) \sim N_3(v), \tag{7.11}$$

where $M_3(u)$ and $N_2(v)$ are N-functions complementary to $N_3(v)$ and $M_2(u)$, respectively.

It follows from (7.10) and (7.11) that $M(u) \sim M_3(u)$. This means that $N(v) \sim N_3(v)$. It follows, again in virtue of (7.10), that $N(v) \sim N_1(v)$. ★

3. *The class* $\mathfrak{N}$. By $\mathfrak{N}$ we shall denote the class of functions which, for large values of the argument, are continuous, non-negative and satisfy the conditions

$$\lim_{u\to\infty} \frac{f[u + \delta(u)]}{f(u)} = \text{constant} > 0 \quad (f \in \mathfrak{N}) \tag{7.12}$$

when

$$\lim_{u\to\infty} \frac{\delta(u)}{u} = d > -1. \tag{7.13}$$

The functions $|u|^\gamma$ for arbitrary γ, $\ln |u|$, and so on, can serve as examples of functions in the class $\mathfrak{N}$. For the first of these, we have that

$$\lim_{u\to\infty} \frac{f[u + \delta(u)]}{f(u)} = \lim_{u\to\infty} \left[1 + \frac{\delta(u)}{u}\right]^\gamma = (1 + d)^\gamma > 0$$

and for the second that

$$\lim_{u\to\infty} \frac{f[u + \delta(u)]}{f(u)} = \lim_{u\to\infty} \left\{1 + \frac{\ln\left[1 + \dfrac{\delta(u)}{u}\right]}{\ln u}\right\} = 1.$$

When the function $f(x)$ is in the class $\mathfrak{N}$, then $1/f(u)$ is also in $\mathfrak{N}$; if the two functions $f_1(u)$ and $f_2(u)$ are in the class $\mathfrak{N}$, then their product $f_1(u)f_2(u)$ is also in $\mathfrak{N}$; if $\lim_{u\to\infty} f_2(u) = \infty$, then the superposition $f_1[f_2(u)]$ is in $\mathfrak{N}$ if the functions $f_1(u)$ and $f_2(u)$ are in $\mathfrak{N}$.

In the proof, we need only the last assertion. Suppose the function $\delta(u)$ satisfies condition (7.13) and that

$$\lim_{u\to\infty} \frac{f_2[u + \delta(u)]}{f_2(u)} = \gamma > 0.$$

Then the function $\delta_1(u)$, defined by the equality

$$\delta_1[f_2(u)] = f_2[u + \delta(u)] - f_2(u),$$

satisfies the condition

$$\lim_{v\to\infty} \frac{\delta_1(v)}{v} = \lim_{u\to\infty} \frac{\delta_1[f_2(u)]}{f_2(u)} = \gamma - 1 > -1,$$

from which it follows that

$$\lim_{u\to\infty} \frac{f_1\{f_2[u + \delta(u)]\}}{f_1[f_2(u)]} = \lim_{u\to\infty} \frac{f_1\{f_2(u) + \delta_1[f_2(u)]\}}{f_1[f_2(u)]} =$$

$$= \lim_{v\to\infty} \frac{f_1[v + \delta_1(v)]}{f_1(v)} = \text{constant} > 0.$$

This means that $f_1[f_2(u)] \in \mathfrak{N}$.

It follows from the properties of the class $\mathfrak{N}$ described above that the function

$$f(u) = u^{\gamma_1}(\ln u)^{\gamma_2}(\ln \ln u)^{\gamma_3} \cdots (\ln \ln \cdots \ln u)^{\gamma_n}, \qquad (7.14)$$

considered for large values of the argument, belongs to $\mathfrak{N}$. In (7.14), the γ_i are arbitrary numbers.

LEMMA 7.3. *Functions from the class $\mathfrak{N}$ which are monotonic for large values of u possess the property that*

$$\lim_{u\to\infty} \frac{\ln f(u)}{u} = 0. \tag{7.15}$$

PROOF. We first consider the case of an increasing function $f(u)$. It follows from (7.12), if we set $\delta(u) = u$ in this condition, that, for u, which are larger than some $u_0 > 0$, we have: $f(2u) \leqslant kf(u)$, where k is some positive number. Suppose $2^n u_0 < u < 2^{n+1}u_0$. Then

$$f(u) \leqslant f(2^{n+1}u_0) \leqslant k^{n+1}f(u_0) \leqslant kf(u_0)2^{n\ln_k 2} \leqslant kf(u_0)\left(\frac{u}{u_0}\right)^{\ln_k 2}.$$

Consequently, $\ln f(u) \leqslant \ln [kf(u_0)u^{-\ln_k 2}] + \ln_k 2 \cdot \ln u$, from which (7.15) does indeed follow.

Now suppose the function $f(u)$ increases monotonically. Then the function $f_1(u) = 1/f(u)$, which also belongs to the class $\mathfrak{N}$, increases monotonically. It follows from the assertion already proved for monotonically increasing functions that

$$\lim_{u\to\infty} \frac{\ln f(u)}{u} = -\lim_{u\to\infty} \frac{\ln f_1(u)}{u} = 0. \ \bigstar$$

We note that the function (7.14) is monotonic for large values of u — a fact which is easily verified.

LEMMA 7.4. *Suppose the function $R(u) \in \mathfrak{M}$ is representable in the form $R(u) = f(\ln u)$, where $f(u) \in \mathfrak{N}$. Then for $\alpha > 1$, we have that*

$$\lim_{v\to\infty} \frac{R\left\{\left[\frac{v}{R(v)}\right]^{\alpha-1}\right\}}{R(v)} = \text{constant} > 0.$$

PROOF. Let $\delta(u) = (\alpha - 2)u - (\alpha - 1)\ln f(u)$. In virtue of the preceding lemma, we have that

$$\lim_{u\to\infty} \frac{\delta(u)}{u} = \alpha - 2 > -1.$$

Since

$$\frac{R\left\{\left[\frac{v}{R(v)}\right]^{\alpha-1}\right\}}{R(v)} = \frac{f[(\alpha-1)\ln v - (\alpha-1)\ln R(v)]}{f(\ln v)} = \\ = \frac{f[\ln v + \delta(\ln v)]}{f(\ln v)},$$

we have that

$$\lim_{v\to\infty} \frac{R\left\{\left[\frac{v}{R(v)}\right]^{\alpha-1}\right\}}{R(v)} = \lim_{u\to\infty} \frac{f[u+\delta(u)]}{f(u)} = \text{constant} > 0. \ \bigstar$$

4. *Theorem on the complementary function.*

THEOREM 7.1. *Suppose $R(u)$ belongs to the class $\mathfrak{M}$ and that it is representable in the form $R(u) = f(\ln u)$, where $f(u)$ is a monotonic function from the class $\mathfrak{N}$. Let an N-function $M(u)$ be prescribed such that p.p. $M(u) = (u^\alpha/\alpha)R(u)$ $(\alpha > 1)$. Finally, suppose the function $(v^\beta/\beta)R^{1-\beta}(v)(1/\alpha + 1/\beta = 1)$ is the principal part of some N-function $N_1(v)$. Then*

$$N(v) \sim N_1(v). \tag{7.16}$$

PROOF. It follows from Lemma 7.4 that condition (7.9) is satisfied. Lemma 7.2 implies (7.16). ★

Let us turn to the consideration of the N-function $M(u)$ pointed out at the beginning of this section:

$$\text{p.p. } M(u) = \frac{u^\alpha}{\alpha}(\ln u)^{\gamma_1}(\ln\ln u)^{\gamma_2}\cdots(\ln\ln\cdots\ln u)^{\gamma_n} \tag{7.1}$$

(where $\alpha > 1$ and the γ_i are arbitrary numbers). This function can be represented in the form p.p. $M(u) = (u^\alpha/\alpha)f(\ln u)$, where $f(u)$ is defined by formula (7.14). Thus, the function $M(u)$ satisfies the condition of Theorem 7.1. We set

$$\text{p.p. } N_1(v) = \frac{v^\beta}{\beta}[(\ln v)^{-\gamma_1}(\ln\ln v)^{-\gamma_2}\cdots(\ln\ln\cdots\ln v)^{-\gamma_n}]^{\beta-1} \\ \left(\frac{1}{\alpha}+\frac{1}{\beta}=1\right). \tag{7.17}$$

Theorem 7.1 implies the next theorem.

THEOREM 7.2. *The N-function* $N_1(v)$ *with principal part* (7.17) *is equivalent to the function complementary to the N-function* $M(u)$ *with principal part* (7.1).

CHAPTER II

ORLICZ SPACES

§ 8. Orlicz classes

1. *Definition.* In the remainder of this book, G will denote a bounded closed set in a finite-dimensional Euclidean space, on which we consider the usual Lebesgue measure. We note that the majority of the assertions and constructions introduced below also retain their validity for the case when we consider an abstract set with finite continuous measure (where by continuity of a measure we understand the existence of a subset A of every set B such that the measure of A equals half the measure of B).

Let $M(u)$ be an N-function. We shall denote by $L_M(G)$ the class of those real-valued functions, defined on G, for which

$$\rho(u; M) = \int_G M[u(x)]\,dx < \infty.$$

In this connection, functions which differ only on a set of measure zero will not be considered distinct. The notation $\rho(u; M)$ will be used frequently in the sequel.

The classes $L_M(G)$ are called *Orlicz classes*. In situations where this cannot give rise to any misunderstanding, we shall write L_M instead of $L_M(G)$.

All bounded functions, but not all summable functions, belong to the class L_M. It is easily seen that every function in the class L_M is summable.

We note that *every function $u(x)$ which is summable on G belongs to some Orlicz class.* To prove this assertion, we consider the sets $G_n = G\{n - 1 \leqslant |u(x)| < n\}$. It is clear that

$$\sum_{n=1}^{\infty} n \operatorname{mes} G_n \leqslant \int_G |u(x)|\,dx + \operatorname{mes} G < \infty.$$

As is known, one can construct an indefinitely increasing se-

quence $\{\alpha_n\}$ such that we also have

$$\sum_{n=1}^{\infty} \alpha_n n \operatorname{mes} G_n < \infty. \tag{8.1}$$

We set

$$p(t) = \begin{cases} t & \text{if } 0 \leqslant t < 1, \\ \alpha_n & \text{if } n \leqslant t < n+1 \quad (n = 1, 2, \cdots). \end{cases}$$

The function $p(t)$ possesses all the properties required in order that

$$M(u) = \int_0^{|u|} p(t)\, dt$$

be an N-function. Since

$$M(n) = \int_0^n p(t)\, dt \leqslant \alpha_n n,$$

we have, in virtue of (8.1), that

$$\int_G M[u(x)]\, dx = \sum_{n=1}^{\infty} \int_{G_n} M[u(x)]\, dx \leqslant \sum_{n=1}^{\infty} M(n) \operatorname{mes} G_n \leqslant$$

$$\leqslant \sum_{n=1}^{\infty} \alpha_n n \operatorname{mes} G_n < \infty.$$

This means that $u(x) \in L_M$.

The assertion just proved means that the space L of functions which are summable on G is the union of all the Orlicz classes.

Further on, we shall use a stronger assertion. It turns out that for every summable function $u(x)$ one can find an N-function $Q(u)$, satisfying the Δ'-condition, such that

$$\int_G Q\{Q[u(x)]\}\, dx < \infty.$$

As we have already shown, there exists an N-function $M(u)$

such that

$$\int_G M[u(x)]\,dx < \infty.$$

The function $M(u)$ can be represented in the form of a composition, $M(u) = Q_1[Q_2(u)]$. We consider the N-function $P(v) = e^{P_1(v)+P_2(v)} - 1$, where $P_1(v)$ and $P_2(v)$ are N-functions which are complementary to $Q_1(u)$ and $Q_2(u)$, respectively. The function $P(v)$ satisfies the Δ^2-condition, where $P_1(v) < P(v)$, $P_2(v) < P(v)$ for large values of v. In virtue of Theorem 6.6, the N-function $Q(u)$ which is complementary to $P(v)$ satisfies the Δ'-condition and for large values of the argument, in virtue of Theorem 2.1, we have that $Q(u) < Q_1(u)$, $Q(u) < Q_2(u)$. Therefore

$$\int_G Q\{Q[(u)]\}\,dx < a + \int_G Q_1\{Q_2[u(x)]\}\,dx < \infty.$$

2. *The Jensen integral inequality.* Let $u(x) \in L_M$; then the inequality

$$M\left\{\frac{\int_G u(x)\,dx}{\operatorname{mes} G}\right\} \leqslant \frac{\int_G M[u(x)]\,dx}{\operatorname{mes} G} \tag{8.2}$$

holds; inequality (8.2) will be called the *Jensen integral inequality.* We note that a more general relation is frequently called Jensen's inequality (see, for example, ZIGMUND [1]).

We shall first consider the case when $u(x)$ is a continuous function. Suppose $\varepsilon > 0$ is an arbitrary prescribed number. The set G can be decomposed into n subsets G_i such that $\operatorname{mes} G_i = (\operatorname{mes} G)/n$ $(i = 1, 2, \cdots, n)$,

$$\left| M\left(\int_G \frac{u(x)}{\operatorname{mes} G}\,dx \right) - M\left(\sum_{i=1}^{n} u(x_i)\,\frac{1}{n} \right) \right| < \varepsilon,$$

$$\left| \frac{\int_G M[u(x)]\,dx}{\operatorname{mes} G} - \sum_{i=1}^{n} \frac{M[u(x_i)]}{n} \right| < \varepsilon,$$

where x_i is some point in the set G_i. From the indicated inequalities

and (1.3), it follows that

$$M\left(\frac{\int_G u(x)\,dx}{\operatorname{mes} G}\right) \leqslant M\left(\sum_{i=1}^{n} u(x_i)\frac{1}{n}\right) + \varepsilon \leqslant \frac{\sum_{i=1}^{n} M[u(x_i)]}{n} + \varepsilon \leqslant$$

$$\leqslant \frac{\int_G M[u(x)]\,dx}{\operatorname{mes} G} + 2\varepsilon.$$

Hence, inequality (8.2) holds, inasmuch as ε is arbitrary.

Inequality (8.2) is obtained for the case of an arbitrary function in L_M by passage to the limit, starting with this same inequality for continuous functions.

3. *The comparison of classes.* Orlicz classes L_{M_1} and L_{M_2} which are determined by distinct N-functions $M_1(u)$ and $M_2(u)$ are, generally speaking, distinct.

THEOREM 8.1. *The inclusion*

$$L_{M_1} \subset L_{M_2} \tag{8.3}$$

holds if, and only if, there exist positive constants u_0 and a such that

$$M_2(u) \leqslant aM_1(u) \quad (u \geqslant u_0). \tag{8.4}$$

PROOF. The sufficiency of condition (8.4) is obvious: for an arbitrary function $u(x) \in L_{M_1}$, we have that

$$\rho(u; M_2) = \int_G M_2[u(x)]\,dx \leqslant$$

$$\leqslant M_2(u_0) \operatorname{mes} G + a\int_G M_1[u(x)]\,dx < \infty.$$

Let us assume that condition (8.4) is not satisfied. Then an indefinitely increasing sequence of numbers u_n can be found such that

$$M_2(u_n) > 2^n M_1(u_n) \quad (n = 1, 2, \cdots). \tag{8.5}$$

We subdivide the set G into disjoint subsets G_n such that

$$\operatorname{mes} G_n = \frac{M_1(u_1) \operatorname{mes} G}{2^n M_1(u_n)} \quad (n = 1, 2, \cdots),$$

and we set

$$u(x) = \begin{cases} u_n & \text{for } x \in G_n, \\ 0 & \text{for } x \bar{\in} \bigcup_{n=1}^{\infty} G_n. \end{cases}$$

The function $u(x)$ is in the class L_{M_1} since

$$\int_G M_1[u(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M_1[u(x)]\,dx = \sum_{n=1}^{\infty} M_1(u_n) \text{ mes } G_n = \sum_{n=1}^{\infty} \frac{M_1(u_1) \text{ mes } G}{2^n} < \infty;$$

but $u(x) \bar{\in} L_{M_2}$ since, in virtue of (8.5), we have that

$$\int_G M_2[u(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M_2[u(x)]\,dx = \sum_{n=1}^{\infty} M_2(u_n) \text{ mes } G_n \geqslant \sum_{n=1}^{\infty} M_1(u_1) \text{ mes } G = \infty. \ \bigstar$$

It follows from this theorem that two functions $M_1(u)$ and $M_2(u)$ determine the same Orlicz class if, and only if, there exist positive constants a, b and u_0 such that

$$aM_2(u) \leqslant M_1(u) \leqslant bM_2(u) \quad (u \geqslant u_0). \tag{8.6}$$

4. *The structure of Orlicz classes.* It follows from Jensen's inequality that the Orlicz class L_M is a convex set: whenever the class L_M contains the two functions $u_1(x)$ and $u_2(x)$ then it also contains the entire "segment" $u_\alpha(x) = \alpha u_1(x) + (1 - \alpha)u_2(x)$ $(0 \leqslant \alpha \leqslant 1)$. In fact, if $u_1(x)$, $u_2(x) \in L_M$, then

$$\rho(u_\alpha; M) = \int_G M[\alpha u_1(x) + (1 - \alpha)u_2(x)]\,dx \leqslant \alpha\rho(u_1; M) + (1 - \alpha)\rho(u_2; M) < \infty.$$

THEOREM 8.2. *The class L_M is a linear set if, and only if, the N-function $M(u)$ satisfies the Δ_2-condition.*

PROOF. Suppose the N-function $M(u)$ satisfies the Δ_2-condition. Then for arbitrary l there exist constants $k(l)$ and u_0 such that $M(lu) \leqslant k(l)M(u)$ for $u \geqslant u_0$. In virtue of Theorem 8.1, it follows from the last inequality that the function $lu(x)$ belongs to the class L_M whenever the function $u(x)$ is in L_M.

Let $u_1(x)$, $u_2(x)$ be in L_M. Then, in virtue of what we have already proved and inequality (1.1), we have that

$$\int_G M[\alpha u_1(x) + \beta u_2(x)]\,dx \leqslant$$

$$\leqslant \tfrac{1}{2}\int_G M[2\alpha u_1(x)]\,dx + \tfrac{1}{2}\int_G M[2\beta u_2(x)]\,dx < \infty$$

for arbitrary α and β.

This proves the sufficiency of the condition of the theorem.

Now, assume that L_M is a linear set. This signifies, in particular, that $2u(x)$ belongs to L_M whenever $u(x) \in L_M$, i.e. that $L_M \subset L_{M_1}$, where the N-function $M(2u)$ is denoted by $M_1(u)$. In virtue of Theorem 8.1, it follows from the last inclusion that there exist constants a and u_0 such that

$$M_1(u) = M(2u) \leqslant aM(u) \text{ for } u \geqslant u_0. \bigstar$$

We shall now consider in more detail the case when the N-function $M(u)$ does not satisfy the Δ_2-condition. Let $u(x) \in L_M$. The totality of functions $u_\beta(x) = \beta u(x)$ $(0 \leqslant \beta < \infty)$ will be called the "ray" passing through $u(x)$. If the function $u(x)$ is bounded, then it is clear that the entire "ray" $u_\beta(x)$ belongs to L_M. In virtue of Theorem 8.2, there also are functions for which part of the "ray" belongs to L_M but the entire "ray" does not belong to this class. We denote by β_0 the number such that $\beta u(x) \in L_M$ for $\beta < \beta_0$ and $\beta u(x) \bar{\in} L_M$ for $\beta > \beta_0$. The question arises whether or not the function $\beta_0 u(x)$ belongs to the class. It turns out that both cases are possible.

Suppose the N-function $M(u)$ does not satisfy the Δ_2-condition. Then a monotonically increasing sequence of numbers u_n $(n = 1, 2, \cdots)$, which tend to infinity, can be found such that $M(u_1) > 1$ and

$$M\left[\left(1 + \frac{1}{n}\right)u_n\right] > 2^n M(u_n) \quad (n = 1, 2, \cdots). \tag{8.7}$$

Let G_n be disjoint subsets of G such that

$$\operatorname{mes} G_n = \frac{\operatorname{mes} G}{2^n M(u_n)} \quad (n = 1, 2, \cdots). \tag{8.8}$$

We define the function $u_*(x)$ by means of the equalities

$$u_*(x) = \begin{cases} u_n \text{ if } x \in G_n \quad (n = 1, 2, \cdots), \\ 0 \quad \text{if } x \bar{\in} \bigcup_{n=1}^{\infty} G_n. \end{cases}$$

The function $u_*(x)$ belongs to L_M since

$$\rho(u_*; M) = \int_G M[u_*(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M[u_*(x)]\,dx =$$

$$= \sum_{n=1}^{\infty} M(u_n) \text{ mes } G_n < \infty.$$

In virtue of (1.17), the functions $\beta u_*(x)$, with $\beta < 1$, also belong to L_M. We shall show that $\beta u_*(x)$ does not belong to L_M when β is > 1. In fact, for $1 + (1/n) < \beta$, we have, in virtue of (8.7) and (8.8), that

$$\int_{G_n} M[\beta u_*(x)]\,dx = M[\beta u_n] \text{ mes } G_n >$$

$$> M\left[\left(1 + \frac{1}{n}\right) u_n\right] \text{mes } G_n > \text{mes } G,$$

from which it follows that

$$\rho(\beta u_*; M) = \int_G M[\beta u_*(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M[\beta u_*(x)]\,dx \geqslant$$

$$\geqslant \sum_{1+(1/n)<\beta} \int_{G_n} M[\beta u_*(x)]\,dx = \infty.$$

We now construct a function $u^*(x)$ such that $\beta u^*(x) \in L_M$ for $\beta < 1$ but $\beta u^*(x) \bar{\in} L_M$ for $\beta \geqslant 1$. To this end, we set

$$u^*(x) = \begin{cases} \left(1 + \frac{1}{n}\right) u_n \text{ if } x \in G_n \quad (n = 1, 2, \cdots), \\ 0 \qquad \text{if } x \bar{\in} \bigcup_{n=1}^{\infty} G_n. \end{cases}$$

In virtue of (8.7) and (8.8), we have, for $\beta \geqslant 1$, that

$$\rho(\beta u^*; M) = \int_G M[\beta u^*(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M[\beta u^*(x)]\,dx \geqslant$$

$$\geqslant \sum_{n=1}^{\infty} M\left[\left(1 + \frac{1}{n}\right) u_n\right] \operatorname{mes} G_n \geqslant \sum_{n=1}^{\infty} 2^n M(u_n) \operatorname{mes} G_n = \infty.$$

But if $\beta < 1$, then, for $\beta[1 + (1/n)] < 1$, we have

$$\int_{G_n} M[\beta u^*(x)]\,dx = M\left[\beta\left(1 + \frac{1}{n}\right) u_n\right] \operatorname{mes} G_n \leqslant \frac{\operatorname{mes} G}{2^n},$$

from which it follows that

$$\rho(\beta u^*; M) = \int_G M[\beta u^*(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M[\beta u^*(x)]\,dx < \infty.$$

§ 9. The space L_M^*

1. *The Orlicz norm.* Suppose $M(u)$ and $N(u)$ are mutually complementary N-functions. We shall denote by $L_M^*(G)$ the totality of functions $u(x)$ satisfying the condition

$$(u, v) = \int_G u(x)v(x)\,dx < \infty$$

for all $v(x) \in L_N$. In this connection, functions which differ only on a set of measure zero are not considered to be distinct — the same as was the case in the definition of Orlicz classes. We shall write L_M^* instead of $L_M^*(G)$ in situations where this cannot give rise to any misunderstanding.

It follows from the definition that L_M^* is a linear set.

In virtue of Young's inequality (2.6), for every pair of functions $u(x) \in L_M$, $v(x) \in L_M$, we have that

$$(u, v) = \int_G u(x)v(x)\,dx \leqslant \rho(u; M) + \rho(v; N), \tag{9.1}$$

from which it follows that $L_M \subset L_M^*$.

THEOREM 9.1. *Suppose* $u(x) \in L_M^*$. *Then*

$$\sup_{\rho(v;\,N)\leqslant 1} |(u, v)| = \sup_{\rho(v;\,N)\leqslant 1} \left| \int_G u(x)v(x)\,dx \right| < \infty.$$

PROOF. Let us assume that the assertion in the theorem is not true. Then one can find a function $u_0(x) \in L_M^*$ and a sequence of functions $v_n(x) \in L_N$, $\rho(v_n; N) \leqslant 1$, such that

$$\int_G u_0(x)v_n(x)\,dx > 2^n \quad (n = 1, 2, \cdots). \tag{9.2}$$

We can assume, without loss of generality, that all these functions are positive.

We consider the increasing sequence of functions

$$g_n(x) = \sum_{k=1}^{n} \frac{1}{2^k} v_k(x) \quad (n = 1, 2, \cdots).$$

It is easily seen that in virtue of the convexity of the N-function $N(v)$ we have

$$\rho(g_n; N) \leqslant \sum_{k=1}^{n} \rho(v_k; N) < 1$$

and also, in virtue of (9.2), that

$$\int_G u_0(x)g_n(x)\,dx \geqslant n \quad (n = 1, 2, \cdots). \tag{9.3}$$

The monotonic sequence of functions $g_n(x)$ converges almost everywhere on G to the function

$$g(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} v_k(x).$$

Since the sequence of functions $N[g_n(x)]$ also increases monotonically, passing to the limit as $n \to \infty$ in the inequality

$$\int_G N[g_n(x)]\,dx < 1,$$

we obtain, in virtue of Levi's theorem, that

$$\int_G N[g(x)]\,dx = \lim_{n\to\infty} \int_G N[g_n(x)]\,dx \leqslant 1.$$

[For Levi's theorem, see NATANSON [1]. LEVI'S THEOREM. — *If a monotonically increasing sequence of measurable functions* $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ *converges almost everywhere on* G *to the function* $\varphi(x)$, *then* $\lim_{n\to\infty} \int_G \varphi_n(x)\,dx = \int_G \varphi(x)\,dx$.]

Thus, the function $g(x)$ is $\in L_N$.

The monotonic sequence of summable functions $u_0(x)g_n(x)$ $(n = 1, 2, \cdots)$ converges almost everywhere to the function $u_0(x)g(x)$. Therefore, in virtue of the aforementioned Levi's theorem and (9.3), we have that

$$\int_G u_0(x)g(x)\,dx = \lim_{n\to\infty} \int_G u_0(x)g_n(x)\,dx = \infty.$$

But this contradicts the fact that $u_0(x) \in L_M^*$. ★

The assertion just proved allows us to introduce the *Orlicz norm* in the set L_M^* with the aid of the following equality:

$$\|u\|_M = \sup_{\rho(v;\,N)\leqslant 1} \left| \int_G u(x)v(x)\,dx \right|. \tag{9.4}$$

It follows from the definition that the norm (9.4) satisties the usual axioms:

1) $\|u\|_M = 0$ if, and only if, $u(x) = 0$ almost everywhere;
2) $\|\alpha u\|_M = |\alpha|\,\|u\|_M$;
3) $\|u_1 + u_2\|_M \leqslant \|u_1\|_M + \|u_2\|_M$.

The set L_M^* thus becomes a normed linear space which is called an *Orlicz space.*

We note further one obvious property of the norm. If $u_1(x)$, $u_2(x) \in L_M^*$ and $|u_1(x)| \leqslant |u_2(x)|$ almost everywhere on G, then $\|u_1\|_M \leqslant \|u_2\|_M$.

As an example, we consider the case of the space L_M^* determined by the N-function $M(u) = |u|^\alpha/\alpha$ $(\alpha > 1)$. The N-function $N(v) = |v|^\beta/\beta$ $(1/\alpha + 1/\beta = 1)$ will be complementary to it.

First assume that $u_1(x) \in L_M^*$ and that

$$\|u_1\|_\alpha = \left(\int_G |u_1(x)|^\alpha dx \right)^{1/\alpha} = 1. \tag{9.5}$$

In virtue of the known Hölder inequality, we have, for arbitrary function $v(x) \in L_N$ satisfying the condition $\rho(v; N) \leqslant 1$, that

$$\left| \int_G u_1(x)v(x)\,dx \right| \leqslant \left(\int_G |u_1(x)|^\alpha dx \right)^{1/\alpha} \left(\int_G |v(x)|^\beta dx \right)^{1/\beta} \leqslant \beta^{1/\beta},$$

so that

$$\|u_1\|_M = \sup_{\rho(v;\,N)\leqslant 1} \left| \int_G u_1(x)v(x)\,dx \right| \leqslant \beta^{1/\beta}. \tag{9.6}$$

On the other hand, for the function $v_0(x) = \beta^{1/\beta}|u_1(x)|^{\alpha-1} \operatorname{sgn} u_1(x)$, which satisfies the condition $\rho(v_0; N) = 1$, we have that

$$\int_G u_1(x)v_0(x)\,dx = \beta^{1/\beta} \int_G |u_1(x)|^\alpha dx = \beta^{1/\beta}.$$

From this and (9.6) it follows that

$$\|u_1\|_M = \beta^{1/\beta}.$$

Suppose now that $u(x)$ is an arbitrary function in L_M^*. Condition (9.5) is satisfied for the function $u_1(x) = u(x)/\|u\|_\alpha$. Therefore

$$\|u\|_M = \beta^{1/\beta} \left(\int_G |u(x)|^\alpha dx \right)^{1/\alpha}. \tag{9.7}$$

Thus, the Orlicz norm defined in the space L_M^* differs from the usual norm in the space L^α by a constant multiplier.

2. *Completeness.*

THEOREM 9.2. *An Orlicz space is complete.*

PROOF. Suppose the sequence of functions $u_n(x) \in L_M^*$ $(n = 1, 2, \ldots)$ is convergent in itself, i.e. that

$$\lim_{n,\,m\to\infty} \|u_n - u_m\|_M = 0. \tag{9.8}$$

This means that, for an arbitrary function $v(x) \in L_M$ which satisfies

the condition $\rho(v; N) \leqslant 1$, we have

$$\lim_{n,m\to\infty} \int_G |u_n(x) - u_m(x)|\,|v(x)|\,dx = 0.$$

It follows from the last relation that the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ converges in measure. Then, it contains a subsequence $u_{n_k}(x)$ $(k = 1, 2, \cdots)$ which converges almost everywhere to some function $u_0(x)$.

Let $\varepsilon > 0$ be an arbitrary prescribed number. In virtue of (9.8), a $k(\varepsilon)$ can be found such that for all k, $k + p > k(\varepsilon)$, we have

$$\int_G |u_{n_{k+p}}(x) - u_{n_k}(x)|\,|v(x)|\,dx < \varepsilon \qquad (*)$$

for all $v(x) \in L_N$ which satisfy the condition $\rho(v; N) \leqslant 1$. Passing to the limit in the inequality (*) as $p \to \infty$, we obtain, in virtue of the Fatou theorem, that

$$\int_G |u_0(x) - u_{n_k}(x)|\,|v(x)|\,dx \leqslant \varepsilon \qquad (**)$$

for all $v(x) \in L_N$ which satisfy the condition $\rho(v; N) \leqslant 1$. [For Fatou's theorem, see NATANSON [1]. FATOU'S THEOREM. — *If the sequence of non-negative measurable functions* $\varphi_1(x), \varphi_2(x), \cdots$ *converges almost everywhere on* G *to the function* $\varphi(x)$, *then*

$$\int_G \varphi(x)\,dx \leqslant \sup_n \left\{ \int_G \varphi_n(x)\,dx \right\}.]$$

It follows from inequality (**), firstly, that $u_0(x) - u_{n_k}(x) \in L_M^*$. Consequently, we also have $u_0(x) \in L_M^*$; secondly, it follows from (**) that

$$\|u_0 - u_{n_k}\|_M \leqslant \varepsilon,$$

i.e. that the subsequence u_{n_k} $(k = 1, 2, \cdots)$ converges in norm to $u_0(x)$. Since $u_{n_k}(x)$ $(k = 1, 2, \cdots)$ is a subsequence of a convergent-in-itself sequence, the initial sequence $u_n(x)$ $(n = 1, 2, \cdots)$ also converges to the function $u_0(x)$. ★

Thus, an Orlicz space is a Banach space.

3. *Norm of the characteristic function.* The characteristic function of the set $\mathscr{E} \subset G$ will be denoted by $\kappa(x; \mathscr{E})$.

Let $v(x)$ be a function in L_N such that $\rho(v; N) \leqslant 1$. Then, according to the Jensen integral inequality (8.2), we have that

$$N\left(\frac{1}{\text{mes } \mathscr{E}} \int_{\mathscr{E}} v(x)\,dx\right) \leqslant \frac{1}{\text{mes } \mathscr{E}} \int_{\mathscr{E}} N[v(x)]\,dx \leqslant \frac{1}{\text{mes } \mathscr{E}},$$

from which it follows that

$$\int_{\mathscr{E}} v(x)\,dx \leqslant \text{mes } \mathscr{E} N^{-1}\left(\frac{1}{\text{mes } \mathscr{E}}\right), \tag{9.9}$$

where $N^{-1}(u)$ is the inverse function of $N(v)$.

For the function $v_0(x) = N^{-1}(1/\text{mes } \mathscr{E})\kappa(x; \mathscr{E})$, satisfying the condition $\rho(v_0; N) = 1$, we have

$$\int_{\mathscr{E}} v_0(x)\,dx = \text{mes } \mathscr{E} N^{-1}\left(\frac{1}{\text{mes } \mathscr{E}}\right). \tag{9.10}$$

By the definition of the norm, we have that

$$\|\kappa(x; \mathscr{E})\|_M = \sup_{\rho(v; M) \leqslant 1} \left| \int_G \kappa(x; \mathscr{E}) v(x)\,dx \right| =$$

$$= \sup_{\rho(v; N) \leqslant 1} \left| \int_{\mathscr{E}} v(x)\,dx \right|,$$

from which, in virtue of (9.9) and (9.10), we obtain the formula for the norm of the characteristic function:

$$\|\kappa(x; \mathscr{E})\|_M = \text{mes } \mathscr{E} N^{-1}\left(\frac{1}{\text{mes } \mathscr{E}}\right). \tag{9.11}$$

4. *Hölder's inequality.* As we have already remarked, the class L_M is contained in the space L_M^*. In this connection, in virtue of (9.1), for every function $u(x) \in L_M$, we have that

$$\|u\|_M = \sup_{\rho(v; N) \leqslant 1} |(u, v)| \leqslant \rho(u; M) + 1. \tag{9.12}$$

LEMMA 9.1. *Let $p(u)$ be the right-hand derivative of the N-function $M(u)$. Suppose $u(x) \in L_M^*$ and that $\|u\|_M \leqslant 1$. Then the function $v_0(x) = p(|u(x)|)$ belongs to L_N and $\rho(v_0; N) \leqslant 1$.*

PROOF. We shall first prove that for any function $v(x) \in L_N$ we have

$$\left| \int_G u(x)v(x)dx \right| \leqslant \begin{cases} \|u\|_M & \text{if } \rho(v; N) \leqslant 1, \\ \|u\|_M \rho(v; N) & \text{if } \rho(v; N) > 1. \end{cases} \tag{9.13}$$

The first inequality in (9.13) is obvious. To obtain the second, we notice it follows from (1.17) with $\rho(v; N) > 1$ that

$$N\left[\frac{v(x)}{\rho(v; N)}\right] \leqslant \frac{N[v(x)]}{\rho(v; N)},$$

so that we have

$$\int_G N\left[\frac{v(x)}{\rho(v; N)}\right] dx \leqslant \frac{1}{\rho(v; N)} \int_G N[v(x)]dx = 1.$$

Therefore

$$\left| \int_G u(x) \frac{v(x)}{\rho(v; N)} dx \right| \leqslant \|u\|_M,$$

from which the second inequality in (9.13) follows.

Suppose $\|u\|_M \leqslant 1$. We set

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \leqslant n, \\ 0 & \text{if } |u(x)| > n. \end{cases}$$

Since the functions $u_n(x)$ are bounded, we have that $p(|u_n(x)|) \in L_N$. We shall assume that the assertion in the lemma is not fulfilled. Then an n_0 can be found such that

$$\int_G N[p(|u_{n_0}(x)|)]dx > 1.$$

In virtue of (2.7), we have that

$$N[p(|u_{n_0}(x)|)] < M[u_{n_0}(x)] + N[p(|u_{n_0}(x)|)] = |u_{n_0}(x)|p(|u_{n_0}(x)|).$$

Integrating this inequality and making use of (9.13), we obtain that

$$\int_G N[p(|u_{n_0}(x)|)]\,dx < \int_G |u_{n_0}(x)|p(|u_{n_0}(x)|)\,dx \leqslant$$

$$\leqslant \|u_{n_0}\|_M \int_G N[p(|u_{n_0}(x)|)]\,dx,$$

which contradicts the inequality $\|u_{n_0}\|_M \leqslant \|u\|_M \leqslant 1$. ★

LEMMA 9.2. *Suppose* $\|u\|_M$ *is* $\leqslant 1$. *Then* $u(x) \in L_M$ *and*

$$\rho(u; M) \leqslant \|u\|_M. \tag{9.14}$$

PROOF. We set $v_0(x) = p(|u(x)|)\ \mathrm{sgn}\ u(x)$. In virtue of Lemma 9.1, we have that $\rho(v_0; N) \leqslant 1$. Since, in virtue of (2.7),

$$u(x)v_0(x) = M[u(x)] + N[v_0(x)],$$

we have that

$$\int_G M[u(x)]\,dx \leqslant \int_G M[u(x)]\,dx + \int_G N[v_0(x)]\,dx =$$

$$= \int_G u(x)v_0(x)\,dx \leqslant \|u\|_M. \text{ ★}$$

The following important inequality follows directly from this lemma:

$$\int_G M\left[\frac{u(x)}{\|u\|_M}\right] dx \leqslant 1. \tag{9.15}$$

THEOREM 9.3. *The inequality*

$$\left|\int_G u(x)v(x)\,dx\right| \leqslant \|u\|_M \|v\|_N \tag{9.16}$$

holds for any pair of functions $u(x) \in L_M^*$, $v(x) \in L_N^*$.

PROOF. In virtue of (9.15), we have that

$$\rho\left(\frac{v}{\|v\|_N};\ N\right) = \int_G N\left[\frac{v(x)}{\|v\|_N}\right] dx \leqslant 1.$$

Therefore

$$\left| \int_G u(x) \frac{v(x)}{\|v\|_N} dx \right| \leqslant \|u\|_M,$$

from which (9.16) follows. ★

Inequality (9.16) will be called *Hölder's inequality*.

5. *The case of the Δ_2-condition.* It follows from inequality (9.15) that the Orlicz space L_M^* is the linear hull of the class L_M. In this connection, in virtue of Theorem 8.2, L_M is a proper subset of L_M^* if the N-function $M(u)$ does not satisfy the Δ_2-condition. In virtue of this same theorem, L_M coincides with L_M^* if $M(u)$ satisfies the Δ_2-condition.

6. *Mean convergence.* We say that a sequence of functions $u_n(x) \in L_M^*$ $(n = 1, 2, \cdots)$ is *mean convergent* to the function $u_0(x) \in L_M^*$ if

$$\lim_{n\to\infty} \int_G M[u_n(x) - u_0(x)]\, dx = 0.$$

It follows from inequality (9.14) that every sequence $u_n(x)$ $(n = 1, 2, \cdots)$ which converges in the norm of L_M^* to some function $u_0(x)$ is also mean convergent to $u_0(x)$. The converse, generally speaking, is not true. In fact, suppose the N-function $M(u)$ does not satisfy the Δ_2-condition. Then there exists a monotonic indefinitely increasing sequence of numbers u_n such that

$$M(2u_n) > \frac{2^n}{\operatorname{mes} G} M(u_n) \quad (n = 1, 2, \cdots).$$

In this connection, we can assume that $M(u_1) > 1$. For each n, we shall construct a system of disjoint sets $G_k^{(n)} \subset G$ $(k = 1, 2, \cdots, n)$ for which

$$\operatorname{mes} G_k^{(n)} = \frac{1}{n} \frac{\operatorname{mes} G}{2^k M(u_k)} \quad (k = 1, 2, \cdots, n),$$

and we set

$$u_n(x) = \begin{cases} u_k \text{ if } x \in G_k^{(n)} & (k = 1, 2, \cdots, n), \\ 0 \text{ if } x \bar{\in} \bigcup_{k=1}^{n} G_k^{(n)}. \end{cases}$$

Then

$$\int_G M[u_n(x)]\,dx = \sum_{k=1}^{n} \int_{G_k^{(n)}} M[u_n(x)]\,dx = \sum_{k=1}^{n} M(u_k)\ \text{mes}\, G_k^{(n)} < \frac{\text{mes}\, G}{n},$$

i.e.

$$\lim_{n\to\infty} \int_G M[u_n(x)]\,dx = 0,$$

from which it follows that the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ is mean convergent to zero. If this sequence were also convergent to zero in norm, then, in virtue of (9.14), the inequality

$$\lim_{n\to\infty} \int_G M[2u_n(x)]\,dx \leqslant \lim_{n\to\infty} \|2u_n\|_M = 0$$

would be satisfied whereas, at the same time, we have that

$$\int_G M[2u_n(x)]\,dx = \sum_{k=1}^{n} \int_{G_k^{(n)}} M[2u_n(x)]\,dx = \sum_{k=1}^{n} M(2u_k)\ \text{meas}\, G_k^{(n)} > 1.$$

The contradiction thus arrived at shows that the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ does not converge in norm.

THEOREM 9.4. *Let the N-function $M(u)$ satisfy the Δ_2-condition. Then convergence in norm is equivalent to mean convergence.*

PROOF. In the proof, we need only the fact that mean convergence implies convergence in norm.

Let $u_n(x) \in L_M^* = L_M$ $(n = 0, 1, \cdots)$ and

$$\lim_{n\to\infty} \int_G M[u_n(x) - u_0(x)]\,dx = 0. \tag{9.17}$$

Let $\varepsilon > 0$ be an arbitrary prescribed number. Assume that $1/2^{k-1} < \varepsilon$.

Since the N-function $M(u)$ satisfies the Δ_2-condition, it follows from (9.17) that

$$\lim_{n\to\infty} \int_G M[2^k(u_n(x) - u_0(x))]\,dx = 0.$$

Let n_0 be a natural number such that for $n \geqslant n_0$ we have

$$\int_G M[2^k(u_n(x) - u_0(x))]\,dx < 1.$$

Then, in virtue of (9.12), for $n \geqslant n_0$, we have that

$$\|2^k(u_n - u_0)\|_M \leqslant \rho(2^k(u_n - u_0); M) + 1 < 2,$$

from which it follows that $\|u_n - u_0\|_M < 1/2^{k-1} < \varepsilon$.

Thus the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ converges in norm to $u_0(x)$. ★

We note that, in the sense of mean convergence, a set of bounded functions is everywhere dense in the class L_M, i.e. for every function $u(x) \in L_M$ we can construct a sequence of bounded functions $u_n(x)$ such that

$$\lim_{n\to\infty} \int_G M[u_n(x) - u(x)]\,dx = 0.$$

The functions $u_n(x)$ can, for example, be defined by the equalities

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \leqslant n, \\ 0 & \text{if } |u(x)| > n. \end{cases}$$

It follows from inequality (9.2) that every set $\mathfrak{N}$ of functions, $\mathfrak{N} \subset L_M$, which are bounded in the mean, i.e. which satisfy the condition

$$\int_G M[u(x)]\,dx \leqslant a \qquad (u(x) \in \mathfrak{N}),$$

will also be bounded in norm, $\|u\|_M \leqslant b$ $(u(x) \in \mathfrak{N})$, where $b = b(a)$ depends only on the constant a. The converse, generally speaking, is not true simply because of the fact that not every function in L_M^* belongs to L_M.

However, if the N-function $M(u)$ satisfies the Δ_2-condition, then the following assertion is valid: *every set* $\mathfrak{N} \subset L_M^* = L_M$, *which is bounded in norm, will also be bounded in the mean.* In fact, suppose $\|u\|_M \leqslant a$ for all $u(x) \in \mathfrak{N}$. Since $M(u)$ satisfies the Δ_2-condition, there exist constants k and u_0 such that $M(au) \leqslant kM(u)$ for $u \geqslant u_0$. Then $M(au) < M(au_0) + kM(u)$ for all u. From this

inequality and (9.15), it follows that

$$\int_G M[u(x)]\,dx \leqslant M(au_0) \text{ mes } G + k \int_G M\left[\frac{u(x)}{a}\right] dx \leqslant$$

$$\leqslant M(au_0) \text{ mes } G + k = b(a),$$

which was to be proved.

7. *The Luxemburg norm.* The set L_M^* can also be transformed into a Banach space with the aid of norms distinct from the norm introduced above.

We shall consider one such norm which was studied in detail in LUXEMBURG [1]. Let

$$\|u\|_{(M)} = \inf k, \tag{9.18}$$

where the infimum is taken over all $k > 0$ such that

$$\rho\left(\frac{u}{k}; M\right) = \int_G M\left[\frac{u(x)}{k}\right] dx \leqslant 1. \tag{9.19}$$

It follows from inequality (9.15) that the inequality

$$\|u\|_{(M)} \leqslant \|u\|_M \tag{9.20}$$

is valid for every function $u(x) \in L_M^*$.

We note that the infimum is attained in (9.18) for those functions $u(x)$ for which $\|u\|_{(M)} > 0$. This follows from inequality (9.19), in which (in virtue of Fatou's theorem) one can pass to the limit as $k \to +\infty$, which limit tends from the right to $\|u\|_M$. Thus, the inequality

$$\rho\left(\frac{u}{\|u\|_{(M)}}; M\right) = \int_G M\left[\frac{u(x)}{\|u\|_{(M)}}\right] dx \leqslant 1 \tag{9.21}$$

holds.

It is quite easily seen that the equality sign occurs in (9.21) if the N-function $M(u)$ satisfies the Δ_2-condition. But if this condition is not satisfied, then functions can be found such that $\rho(u/\|u\|_{(M)}; M) < 1$. It is also easily seen that the equality

$$\int_G M\left[\frac{u(x)}{k_0}\right] dx = 1 \tag{9.22}$$

always implies that $k_0 = \|u\|_{(M)}$.

We shall show that the norm $\|u\|_{(M)}$ satisfies the usual axioms.

1°. Inequality (9.19) is satisfied for arbitrary k if, and only if, $u(x) = 0$ almost everywhere. Therefore, $\|u\|_{(M)} = 0$ if, and only if, $u(x) = 0$ almost everywhere.

2°. The equality $\|\alpha u\|_{(M)} = |\alpha| \|u\|_{(M)}$ follows from the obvious relations

$$\|\alpha u\|_{(M)} = \inf_{\rho(\alpha u/k;\, M) \leqslant 1} k = |\alpha| \inf_{\rho(u/k_1;\, M) \leqslant 1} k_1 = |\alpha| \|u\|_{(M)}.$$

3°. The triangle inequality

$$\|u_1 + u_2\|_{(M)} \leqslant \|u_1\|_{(M)} + \|u_2\|_{(M)}$$

is obvious in the case when the norm of one of the functions $u_1(x)$, $u_2(x)$ is zero. If $\|u_1\|_{(M)} > 0$ and $\|u_2\|_{(M)} > 0$, then, in virtue of (1.2), we have

$$M\left[\frac{u_1(x) + u_2(x)}{\|u_1\|_{(M)} + \|u_2\|_{(M)}}\right] \leqslant \frac{\|u_1\|_{(M)}}{\|u_1\|_{(M)} + \|u_2\|_{(M)}} M\left[\frac{u_1(x)}{\|u_1\|_{(M)}}\right] +$$

$$+ \frac{\|u_2\|_{(M)}}{\|u_1\|_{(M)} + \|u_2\|_{(M)}} M\left[\frac{u_2(x)}{\|u_2\|_{(M)}}\right]$$

and, in virtue of (9.21),

$$\int_G M\left[\frac{u_1(x) + u_2(x)}{\|u_1\|_{(M)} + \|u_2\|_{(M)}}\right] dx \leqslant 1,$$

from which the triangle inequality does indeed follow.

As an example, we shall find $\|\kappa(x; \mathscr{E})\|_{(M)}$ of the characteristic function of the set $\mathscr{E} \subset G$. If mes $\mathscr{E} \neq 0$, then

$$\|\kappa(x; \mathscr{E})\|_{(M)} = \frac{1}{M^{-1}\left(\frac{1}{\text{mes } \mathscr{E}}\right)} \tag{9.23}$$

inasmuch as

$$\int_G M\left[\kappa(x; \mathscr{E}) M^{-1}\left(\frac{1}{\text{mes } \mathscr{E}}\right)\right] dx = 1.$$

THEOREM 9.5. *The unit sphere of the space L_M^* with respect to the norm $\|u\|_{(M)}$ coincides with the set of functions $u(x) \in L_M$ for which $\rho(u; M) \leqslant 1$. Moreover, it follows from $\|u\|_{(M)} \leqslant 1$ thta $\rho(u; M) \leqslant \|u\|_{(M)}$ and it follows from $\|u\|_{(M)} > 1$ that $\rho(u; M) \geqslant \geqslant \|u\|_{(M)}$.*

PROOF. Suppose $\|u\|_{(M)} \leqslant 1$. Then, in virtue of (1.17) and (9.21), we have that

$$\frac{1}{\|u\|_{(M)}} \int_G M[u(x)]\,dx \leqslant \int_G M\left[\frac{u(x)}{\|u\|_{(M)}}\right] dx \leqslant 1,$$

i.e. $\rho(u; M) \leqslant \|u\|_{(M)}$. But if $\|u\|_{(M)} > 1$, then, in virtue of (1.17), we have that

$$\frac{1}{\|u\|_{(M)} - \varepsilon} \int_G M[u(x)]\,dx \geqslant \int_G M\left[\frac{u(x)}{\|u\|_{(M)} - \varepsilon}\right] dx > 1$$

for all sufficiently small $\varepsilon > 0$, from which it follows that $\rho(u; M) \geqslant \|u\|_{(M)}$ inasmuch as ε is arbitrary. ★

We shall show that the norms $\|u\|_M$ and $\|u\|_{(M)}$ are equivalent:

$$\|u\|_{(M)} \leqslant \|u\|_M \leqslant 2\|u\|_{(M)}. \tag{9.24}$$

We have already proved the first of these inequalities. The second follows from (9.12) and (9.21):

$$\left\| \frac{u}{\|u\|_{(M)}} \right\|_M \leqslant \int_G M\left[\frac{u(x)}{\|u\|_{(M)}}\right] dx + 1 \leqslant 2.$$

Still another formula for the definition of the Orlicz norm $\|u\|_M$ follows from Theorem 9.5:

$$\|u\|_M = \sup_{\|v\|_{(N)} \leqslant 1} \left| \int_G u(x)v(x)\,dx \right|. \tag{9.25}$$

From (9.25), there follow, in turn, inequalities which it is natural to call strengthened Hölder inequalities:

$$\left| \int_G u(x)v(x)\,dx \right| \leqslant \|u\|_M \|v\|_{(N)} \qquad (u(x) \in L_M^*,\ v(x) \in L_N^*) \tag{9.26}$$

and

$$\left| \int_G u(x)v(x)\,dx \right| \leqslant \|u\|_{(M)} \|v\|_N \qquad (u(x) \in L_M^*,\ v(x) \in L_N^*). \tag{9.27}$$

§ 10. The space E_M

1. *Definition.* We shall denote the closure in L_M^* of the set of bounded functions by E_M.

As we have already noted, the set of bounded functions is everywhere dense in the Orlicz class L_M in the sense of mean convergence. It follows from Theorem 9.4 that the set of bounded functions is everywhere dense in the Orlicz space $L_M^* = L_M$ provided the Δ_2-condition is satisfied. Thus, if the N-function $M(u)$ satisfies the Δ_2-condition, then the spaces E_M and L_M^* coincide.

In the case when the N-function $M(u)$ does not satisfy the Δ_2-condition, E_M is a proper subset of L_M^*. This follows from the following inclusion:

$$E_M \subset L_M. \tag{10.1}$$

We shall now prove this inclusion. Suppose $u_0(x) \in E_M$ and that $u_1(x)$ is a bounded function such that $\|u_0 - u_1\|_M < \frac{1}{2}$. Then, in virtue of (9.13),

$$\int_G M[2u_0(x) - 2u_1(x)]\,dx \leqslant 2\|u_0 - u_1\|_M < 1.$$

This means that $2u_0(x) - 2u_1(x) \in L_M$. Since bounded functions belong to L_M, it follows from the convexity of the set L_M that the function $u_0(x) = \frac{1}{2}[2u_0(x) - 2u_1(x)] + \frac{1}{2}[2u_1(x)]$ also belongs to L_M.

2. *The separability of* E_M. Suppose $u(x)$ is a bounded function: $|u(x)| \leqslant a$. In virtue of Luzin's theorem, a sequence of continuous functions $u_n(x)$, $|u_n(x)| \leqslant a$, can be found such that the difference $u(x) - u_n(x)$ is different from zero only on a set $G_n \subset G$ whose measure is less than $1/n$. Then, in virtue of formula (9.11), for the norm of the characteristic function we have

$$\|u - u_n\|_M = \sup_{\rho(v;\,N)\leqslant 1} \left| \int_G [u(x) - u_n(x)]v(x)\,dx \right| \leqslant$$

$$\leqslant 2a \sup_{\rho(v;\,N)\leqslant 1} \int_{G_n} |v(x)|\,dx = 2a\|\kappa(x; G_n)\|_M =$$

$$= 2a \operatorname{mes} G_n N^{-1}\left(\frac{1}{\operatorname{mes} G_n}\right) \leqslant \frac{2a}{n} N^{-1}(n),$$

from which it follows that $\lim_{n\to\infty} \|u - u_n\|_M = 0$.

Thus, the set of continuous functions is everywhere dense in the space E_M.

For every function $u(x)$, which is continuous on G, we can find a sequence of polynomials with rational coefficients which converges uniformly to $u(x)$. It is easily seen that such a sequence also converges to $u(x)$ with respect to the norm of any Orlicz space. Consequently, the countable set of polynomials with rational coefficients is everywhere dense in the space E_M.

Thus, the space E_M is separable.

3. *Disposition of the class L_M with respect to the space E_M.* As we have already shown, $E_M \subset L_M$. We shall denote the totality of functions $u(x) \in L_M^*$ for which

$$d(u, E_M) = \inf_{w \in E_M} \|u - w\|_M < r$$

by $\Pi(E_M; r)$. The closure of $\Pi(E_M; r)$ will be denoted by $\overline{\Pi}(E_M; r)$.

The disposition of the class L_M with respect to the space E_M is sufficiently completely described by the following theorem (see Fig. 7).

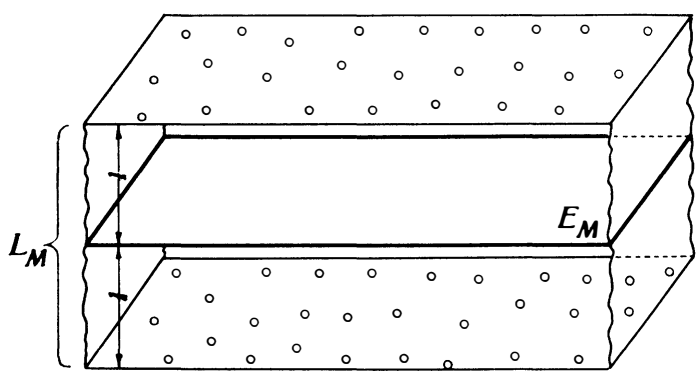

Fig. 7.

THEOREM 10.1. *Suppose the N-function $M(u)$ does not satisfy the Δ_2-condition. Then*

$$\Pi(E_M; 1) \subset L_M \subset \overline{\Pi}(E_M; 1). \tag{10.2}$$

In this connection, $\Pi(E_M; 1)$ is a proper subset of L_M and L_M is a proper subset of $\overline{\Pi}(E_M; 1)$.

PROOF. We shall first prove that every function $u_0(x) \in E_M$ is contained in L_M together with an open spherical neighborhood of radius 1. This will prove the first inclusion in (10.2).

Let $u(x) \in L_M^*$ and $\|u - u_0\|_M < 1$. Then a number $\alpha > 0$ can be found such that $\|u - u_0\|_M < 1 - \alpha$. Since the set E_M is linear, the function $(1/\alpha)u_0(x)$ is in E_M and, in virtue of (10.1), we have that $(1/\alpha)u_0(x) \in L_M$. Since $\|(u - u_0)/(1 - \alpha)\|_M < 1$, we have, in virtue of (9.14), that $\{u(x) - u_0(x)\}/(1 - \alpha) \in L_M$. It follows from the convexity of the set L_M that the function

$$u(x) = (1 - \alpha)\left[\frac{u(x) - u_0(x)}{1 - \alpha}\right] + \alpha\frac{u_0(x)}{\alpha}$$

also belongs to L_M.

For the proof of the second inclusion in (10.2), we shall show that every function $u(x) \in L_M$ is at a distance not greater than unity from E_M.

In virtue of the absolute continuity of the integral, for arbitrary $\varepsilon > 0$, a bounded function $u_\varepsilon(x) \in E_M$ can be found such that

$$\int_G M[u(x) - u_\varepsilon(x)]\,dx < \varepsilon.$$

Then, in virtue of inequality (9.12), $\|u - u_\varepsilon\| < 1 + \varepsilon$, from which it follows that

$$d(u, E_M) = \inf_{w \in E_M} \|u - w\|_M \leqslant \|u - u_\varepsilon\| < 1 + \varepsilon.$$

Since ε is arbitrary, we have that $d(u, E_M) \leqslant 1$.

It remains to prove that the inclusions (10.2) are proper.

In subsection 4 of § 8 we constructed a function $u_*(x) \in L_M$ such that $\beta u_*(x) \in L_M$ for $\beta \leqslant 1$ and $\beta u(x) \overline{\in} L_M$ for $\beta > 1$. We shall show that $u_*(x) \overline{\in} \Pi(E_M; 1)$. In fact, if $d(u_*, E_M)$ were less than unity, then a $\beta > 1$ could be found such that

$$d(\beta u_*, E_M) = \inf_{w \in E_M} \|\beta u_* - w\|_M = \beta \inf_{w \in E_M} \|u_* - w\| < 1.$$

From this it would follow, in virtue of the first inclusion in (10.2), that $\beta u_*(x) \in L_M$ contrary to the basic property of the function $u_*(x)$. It is thus proved that $\Pi(E_M; 1)$ is a proper subset of L_M.

In subsection 4 of § 8 we also constructed a function $u^*(x) \overline{\in} L_M$ such that $\beta u^*(x) \in L_M$ for $\beta < 1$ and $\beta u^*(x) \overline{\in} L_M$ for $\beta \geqslant 1$. We

shall show that $u^*(x) \in \overline{\Pi}(E_M; 1)$. This will complete the proof of the theorem.

In fact, $d(u^*, E_M) = 1$ since otherwise a $\beta < 1$ could be found such that $d(\beta u^*, E_M) > 1$ from which it follows, in virtue of the second inclusion in (10.2), that $\beta u^*(x) \bar{\in} L_M$ contrary to the basic property of the function $u^*(x)$. ★

The second part of the assertion of the theorem signifies that the class L_M is neither an open set nor a closed set in the space L_M^* if the N-function $M(u)$ does not satisfy the Δ_2-condition. We suggest that the reader show in this connection that the set L_M is not complete in the sense of mean convergence.

The following assertion is implied by Theorem 8.2: if the N-function $M(u)$ does not satisfy the Δ_2-condition, then the set of bounded functions is nowhere dense in L_M^* inasmuch as all bounded functions are in E_M.

The space E_M could have been defined as the maximal linear subspace of the space L_M^* which is contained in L_M. This follows from the fact that $u(x) \in E_M$ only in case $\lambda u(x) \in L_M$ for all values of λ.

We shall assume that $u(x) \in L_M^*$ and that

$$d(u, E_M) = \inf_{w \in E_M} \|u - w\|_M > 0.$$

Then the function $u(x)$ can be approximated in the L_M^* norm by a bounded function to within $d(u, E_M) + \varepsilon$, where ε is an arbitrary positive number. It is convenient in a number of cases to select the approximating bounded functions in a special way.

LEMMA 10.1. *The equality*

$$\lim_{n \to \infty} \|u - u_n\|_M = d(u, E_M)$$

holds for an arbitrary function $u(x) \in L_M^*$, *where*

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \leqslant n, \\ 0 & \text{if } |u(x)| > n. \end{cases}$$

PROOF. The functions $|u(x) - u_n(x)|$ do not increase as $n \to \infty$. Therefore their norms do not increase and have a limit. Clearly,

$$\lim_{n \to \infty} \|u - u_n\|_M \geqslant d(u, E_M).$$

It remains to prove the reverse inequality.

Let ε be an arbitrary positive number and let

$$\frac{1}{d(u, E_M) + 2\varepsilon} < \alpha < \frac{1}{d(u, E_M) + \varepsilon}.$$

Then

$$d(\alpha u, E_M) = \inf_{w \in E_M} \|\alpha u - w\|_M = \alpha \inf_{w \in E_M} \|u - w\|_M <$$

$$< \frac{d(u, E_M)}{d(u, E_M) + \varepsilon} < 1.$$

Therefore

$$\int_G M[\alpha u(x)]\, dx < \infty.$$

In virtue of the absolute continuity of the integral, an n_0 can be found such that, for $n \geqslant n_0$, we have

$$\int_G M[\alpha u(x) - \alpha u_n(x)]\, dx < \alpha\varepsilon,$$

from which it follows that $\|\alpha u - \alpha u_n\|_M < 1 + \alpha\varepsilon$, i.e. that $\|u - u_n\|_M < (1/\alpha) + \varepsilon < d(u, E_M) + 3\varepsilon$, and, in virtue of the fact that ε is arbitrary, that $\lim_{n\to\infty} \|u - u_n\|_M \leqslant d(u, E_M)$. ★

4. *Necessary conditions for separability of an Orlicz space.* As was shown above, the space E_M is always separable. This means that the space $L_M^* = L_M = E_M$ is also separable if the N-function $M(u)$ satisfies the Δ_2-condition.

THEOREM 10.2. *Suppose the N-function $M(u)$ does not satisfy the Δ_2-condition. Then the space L_M^* is not separable.*

PROOF. We shall assume that L_M^* is separable and let $u_n(x)$ $(n = 1, 2, \cdots)$ be a countable everywhere dense set in L_M^*. In virtue of Luzin's theorem, a set $G_1 \subset G$ of measure zero can be found on which the functions $u_n(x)$ $(n = 1, 2, \cdots)$ are continuous.

We consider the space $L_M^*(G_1)$ and denote the closure, in $L_M^*(G_1)$, of the set of functions bounded on G_1 by $E_M(G_1)$. Then the functions $w_n(x)$ $(n = 1, 2, \cdots)$, which are defined on G_1 and coincide on this set with the corresponding functions $u_n(x)$, belong to $E_M(G_1)$.

In virtue of Theorem 10.1, a function $w(x) \in L_M^*(G_1)$ can be

found such that $d(w, E_M(G_1)) > 1$. We set

$$u(x) = \begin{cases} w(x) & \text{if } x \in G_1, \\ 0 & \text{if } x \in G \setminus G_1. \end{cases}$$

Then

$$\|u - u_n\|_M = \sup_{\rho(v;\, N) \leqslant 1} \left| \int_G [u(x) - u_n(x)] v(x)\, dx \right| \geqslant$$

$$\geqslant \sup_{\rho(v;\, N) \leqslant 1} \left| \int_{G_1} [u(x) - u_n(x)] v(x)\, dx \right| = \|w - w_n\|_M > 1.$$

This means that the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ is not dense in the space L_M^*. We have thus arrived at a contradiction. ★

5. *On the definition of the norm.* Since $E_N \subset L_N$, we have that

$$\sup_{\substack{\rho(v;\, N) \leqslant 1 \\ v \in E_N}} \left| \int_G u(x) v(x)\, dx \right| \leqslant \sup_{\rho(v;\, N) \leqslant 1} \left| \int_G u(x) v(x)\, dx \right| = \|u\|_M. \qquad (10.3)$$

Suppose $\varepsilon > 0$ is an arbitrary prescribed number. Then a function $v_0(x) \in L_N$ can be found, satisfying the condition $\rho(v_0; N) \leqslant 1$, such that

$$\int_G u(x) v_0(x)\, dx \geqslant \|u\|_M - \varepsilon.$$

We set

$$v_n(x) = \begin{cases} v_0(x) & \text{if } |v_0(x)| \leqslant n, \\ 0 & \text{if } |v_0(x)| > n. \end{cases}$$

Clearly, $v_n(x) \in E_N$ and $\rho(v_n; N) \leqslant \rho(v_0; N) \leqslant 1$ $(n = 1, 2, \cdots)$.

It follows from the absolute continuity of the integral that for sufficiently large n,

$$\int_G u(x) v_n(x)\, dx \geqslant \int_G u(x) v_0(x)\, dx - \varepsilon \geqslant \|u\|_M - 2\varepsilon.$$

Consequently,

$$\sup_{\substack{\rho(v;\, N) \leqslant 1 \\ v \in E_N}} \left| \int_G u(x) v(x)\, dx \right| \geqslant \int_G u(x) v_n(x)\, dx \geqslant \|u\|_M - 2\varepsilon.$$

The preceding inequality and (10.3) yield, in view of the fact that ε is arbitrary, a new representation for the norm in the space L_M^*:

$$\|u\|_M = \sup_{\substack{\rho(v;\,N)\leqslant 1 \\ v\in E_N}} \left| \int_G u(x)v(x)\,dx \right|. \tag{10.4}$$

We showed that formula (10.4) defines a norm in an Orlicz space having assumed that $u(x) \in L_M^*$. A line of reasoning, similar to that used in the proof of Theorem 9.1, shows that the finiteness of the integrals $\int_G u(x)v(x)\,dx$ for all $v(x) \in E_N$ implies that the functions $u(x)$ belong to the space L_M^*. The modification of the proof of Theorem 9.1 consists in this that the functions $v_n(x)$ are selected from E_N; the function $g(x)$ also belongs to E_N since the series which defines this function converges in norm and E_N is closed.

6. *The absolute continuity of the norm.* We shall say that the function $u(x) \in L_M^*$ has an *absolutely continuous norm* if for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$\|u\kappa(x;\,\mathscr{E})\|_M = \sup_{\rho(v;\,N)\leqslant 1} \left| \int_{\mathscr{E}} u(x)v(x)\,dx \right| < \varepsilon$$

provided mes $\mathscr{E} < \delta$ ($\mathscr{E} \subset G$).

THEOREM 10.3. *A function $u(x) \in L_M^*$ has an absolutely continuous norm if, and only if, $u(x) \in E_M$.*

PROOF. Let $u(x) \in E_M$. Suppose $\varepsilon > 0$ is an arbitrarily prescribed number. We denote by $u_1(x)$ a bounded function, $|u_1(x)| < a$ ($x \in G$), such that $\|u - u_1\|_M < \varepsilon/2$.

Since the function $vN^{-1}(1/v)$ increases monotonically for $v \geqslant 0$, the equation $\delta N^{-1}(1/\delta) = \varepsilon/(2a)$ has the unique solution $\delta > 0$.

Suppose mes $\mathscr{E} < \delta$ ($\mathscr{E} \subset G$). Then, in virtue of formula (9.11), we have that

$$\|u\kappa(x;\,\mathscr{E})\|_M \leqslant \|u - u_1\|_M + a\|\kappa(x;\,\mathscr{E})\|_M \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + a \text{ mes } \mathscr{E} N^{-1}\left(\frac{1}{\text{mes } \mathscr{E}}\right) \leqslant \frac{\varepsilon}{2} + a\delta N^{-1}\left(\frac{1}{\delta}\right) = \varepsilon$$

for the norm of the characteristic function.

This proves the absolute continuity of the norm of the function $u(x) \in E_M$.

Now, suppose that the norm of the function $u(x) \in L_M^*$ is absolutely continuous. We denote the set $G\{|u(x)| \leqslant n\}$ by G_n. Since the function $u(x)$ is summable, we have that $\lim_{n\to\infty} \operatorname{mes}(G \setminus G_n) = 0$, from which it follows, in virtue of the absolute continuity of the norm, that $\lim_{n\to\infty} \|u - u\kappa(x; G_n)\|_M = 0$. Thus, $u(x)$, as the limit of a sequence of bounded functions, belongs to E_M. ★

It follows from Theorem 10.3 that the space E_M can be defined as the totality of all functions having absolutely continuous norm in L_M^*.

It also follows from Theorem 10.3 that *all functions in the space L_M^* have absolutely continuous norms if, and only if, the N-function $M(u)$ satisfies the Δ_2-condition.*

7. *Calculation of the norm.* The usual formula

$$\|u\|_M = \sup_{\rho(v;\, N)\leqslant 1} \left| \int_G u(x)v(x)\,dx \right|$$

does not allow us to actually carry out the computation of the norm. In this connection, the problem arises of expressing the norm in other forms.

THEOREM 10.4. *Let $u(x) \in L_M^*$ and suppose there exists a positive number k^* such that*

$$\int_G N[p(k^*|u(x)|)]\,dx = 1, \tag{10.5}$$

where $p(u)$ is the right derivative of the N-function $M(u)$. Then

$$\|u\|_M = \int_G p(k^*|u(x)|)\,|u(x)|\,dx. \tag{10.6}$$

PROOF. In virtue of (10.5), we have that

$$\int_G p(k^*|u(x)|)\,|u(x)|\,dx \leqslant \sup_{\rho(v;\, N)\leqslant 1} \left| \int_G u(x)v(x)\,dx \right| = \|u\|_M.$$

On the other hand, in virtue of Young's inequality,

$$\|u\|_M = \frac{1}{k^2} \sup_{\rho(v;\,N)\leqslant 1} \left| \int_G k^* u(x) v(x)\, dx \right| \leqslant \frac{1}{k^*} \left(1 + \int_G M[k^* u(x)]\, dx \right)$$

and, in virtue of (10.5),

$$\|u\|_M \leqslant \frac{1}{k^*} \left(\int_G N[p(k^* |u(x)|)]\, dx + \int_G M[k^* u(x)]\, dx \right).$$

Utilizing (2.7), we obtain that

$$\|u\|_M \leqslant \frac{1}{k^*} \int_G p(k^* |u(x)|) k^* |u(x)|\, dx = \int_G p(k^* |u(x)|)\, |u(x)|\, dx. \bigstar$$

In seeking the number k^*, equation (10.5) can, in virtue of (2.7), be written in the form

$$k^* \int_G |u(x)|\, p(k^* |u(x)|)\, dx - \int_G M[k^* u(x)]\, dx = 1. \tag{10.7}$$

From this it is clear that to calculate the norm, making use of Theorem 10.4, one must know only the N-function $M(u)$ and its derivative $p(u)$.

We note that the constant k^*, for which equality (10.5) is fulfilled, cannot always be found. For example, if the function $p(u)$ is discontinuous, then it is impossible to find a number k^* even for the characteristic function of the set G if $N^{-1}(1/\operatorname{mes} G)$ does not belong to the set of values of the function $p(u)$.

On the other hand, if the function $p(u)$ has intervals of constancy, then the number k^* is not uniquely defined.

If the function $p(u)$ is continuous, then the number k^* can be found for any bounded function. This follows from the fact that the function

$$J(k) = \int_G N[p(k |u(x)|)]\, dx$$

is defined for all $k > 0$ and it is continuous, with $J(0) = 0$, $J(\infty) = \infty$.

Further on (in § 18, subsection 10) we shall show that the number k^* can be found for any function $u(x) \in E_M$ provided simply that the function $p(u)$ is continuous.

Formula (10.6) allows us to actually carry out the calculation of the norm with any degree of accuracy. This calculation reduces to the approximate solution of equation (10.5) with respect to k^* and then to the calculation of the integral (10.6).

We note that Theorem 10.4 enables us, in the case of a continuous function $p(u)$, to obtain an already-known formula for the norm of the characteristic function.

As an example, we shall compute the norm of the function $u(x) = x$ in the space $L_M^*([0, 1])$, where $M(u) = e^{|u|} - |u| - 1$. Since, in this case, $N[p(u)] = ue^u - e^u + 1$, equation (10.5) has the form

$$\int_0^1 (k^* x e^{k^* x} - e^{k^* x} + 1)\, dx = 1.$$

From this we obtain that k^* is the positive root of the equation

$$e^k = \frac{2}{2 - k}. \tag{10.8}$$

The value of the norm is determined by formula (10.6):

$$\|u\|_M = \int_0^1 (e^{k^* x} - 1) x\, dx,$$

from which it follows, in virtue of (10.8), that

$$\|u\|_M = \frac{1}{k^*(2 - k^*)} - \frac{1}{2}.$$

Equation (10.8) can be solved approximately (e.g. by a graphical method — as in Fig. 8). It turns out that $k^* \approx 1.587$. Thus $\|u\|_M \approx 1.027$.

It is convenient to use Theorem 10.4 in the calculation of the norms of bounded functions. We note that for an arbitrary function $u(x) \in L_M^*$ one can construct a sequence of bounded functions $u_n(x)$

such that $\lim_{n\to\infty} \|u_n\|_M = \|u\|_M$. The functions $u_n(x)$ can be defined, for instance, by the equalities

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \leqslant n, \\ 0 & \text{if } |u(x)| > n. \end{cases} \tag{10.9}$$

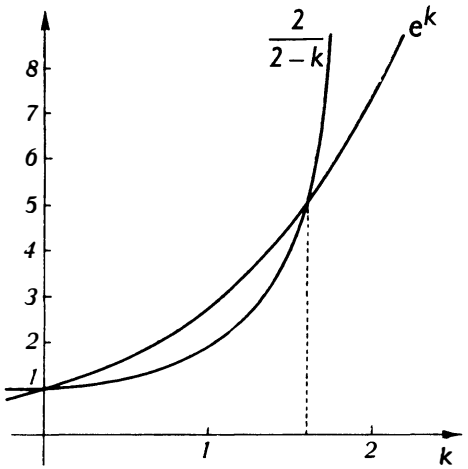

Fig. 8.

In fact, $\|u_1\|_M \leqslant \|u_2\|_M \leqslant \cdots \leqslant \|u_n\|_M \leqslant \cdots \leqslant \|u\|_M$. Let $\varepsilon > 0$ be given number. Then a function $v_0(x)$ can be found such that $\rho(v_0; N) \leqslant 1$ and

$$\int_G |u(x)v_0(x)|\,dx > \|u\|_M - \varepsilon.$$

In virtue of Lebesgue's theorem on passage to the limit under the integral sign, we have that

$$\int_G |u_n(x)v_0(x)|\,dx > \|u\|_M - \varepsilon$$

for sufficiently large n, i.e. $\lim_{n\to\infty} \|u_n\|_M \geqslant \|u\|_M - \varepsilon$. We now have that $\lim_{n\to\infty} \|u_n\|_M \geqslant \|u\|_M$ inasmuch as ε is arbitrary.

8. *Another formula for the norm.* Under the conditions of

Theorem 10.4, we have that

$$\|u\|_M = \frac{1}{k^*}\left(1 + \int_G M[k^* u(x)]\,dx\right).$$

On the other hand,

$$\|u\|_M = \frac{1}{k} \sup_{\rho(v;\,N)\leqslant 1} \left| \int_G ku(x)v(x)\,dx \right| \leqslant$$

$$\leqslant \frac{1}{k}\left(1 + \int_G M[ku(x)]\,dx\right) \qquad (10.10)$$

for arbitrary function $u(x) \in L_M^*$ and any $k > 0$. Therefore, under the conditions of Theorem 10.4, we have that

$$\|u\|_M = \min_{k>0} \frac{1}{k}\left(1 + \int_G M[ku(x)]\,dx\right).$$

It turns out that this formula can be generalized.

THEOREM 10.5. *Let $M(u)$ be an arbitrary N-function where $u(x) \in L_M^*$. Then the formula*

$$\|u\|_M = \inf_{k>0} \frac{1}{k}\left(1 + \int_G M[ku(x)]\,dx\right) \qquad (10.11)$$

is valid.

PROOF. We shall first assume that the function $p(u)$ is continuous. Then the norm of every bounded function can be defined by means of formulas (10.5) and (10.6). It has already been shown that in this case the norm is also defined by formula (10.11).

Let $u(x)$ be an arbitrary function in L_M^*. Suppose the functions $u_n(x)$ are defined by formulas (10.9) and that

$$\|u_n\|_M = \frac{1}{k_n}\left(1 + \int_G M[k_n u_n(x)]\,dx\right), \qquad (10.12)$$

where

$$\int_G N[p(k_n |u_n(x)|)]\,dx = 1.$$

It follows from the last equality that the k_n do not increase. In this connection,

$$\frac{1}{k_n} < \frac{1}{k_n}\left(1 + \int_G M[k_n u_n(x)]\,dx\right) = \|u_n\|_M \leqslant \|u\|_M.$$

Consequently, the sequence k_n converges to some positive number k^*.

Let $\varepsilon > 0$ be a prescribed number. In virtue of (10.12), we have

$$1 + \int_G M[k_n u_n(x)]\,dx = k_n \|u_n\|_M < (k^* + \varepsilon)\,\|u\|_M$$

for sufficiently large n. Passing to the limit in this inequality (which can be done in virtue of Fatou's theorem), we obtain that

$$1 + \int_G M[k^* u(x)]\,dx \leqslant \|u\|_M (k^* + \varepsilon),$$

and, in virtue of the fact that ε is arbitrary, that

$$\frac{1}{k^*}\left(1 + \int_G M[k^* u(x)]\,dx\right) \leqslant \|u\|_M.$$

Equality (10.11) follows from this inequality and (10.10).

We now liberate ourselves from the assumption on the continuity of $p(u)$. Whatever the N-function $M(u)$, one can easily construct an N-function $M_1(u)$ with continuous derivative such that

$$M(u) < M_1(u) < M((1 + \varepsilon)u) \quad (u > 0). \tag{10.13}$$

It follows from these inequalities that $L_M^* = L_{M_1}^*$ and that

$$\|u\|_M \leqslant \|u\|_{M_1} \leqslant (1 + \varepsilon)\,\|u\|_M. \tag{10.14}$$

(A stronger proposition will be proved below in § 13, subsections 1 and 2.) It follows from these same inequalities (10.13) that

$$\inf_{k>0} \frac{1}{k}\left(1 + \int_G M[ku(x)]\,dx\right) \leqslant \inf_{k>0} \frac{1}{k}\left(1 + \int_G M_1[ku(x)]\,dx\right) \leqslant$$

$$\leqslant (1 + \varepsilon) \inf_{k>0} \frac{1}{k}\left(1 + \int_G M[ku(x)]\,dx\right)$$

from which it follows, by what has already been proved, that

$$\frac{1}{1+\varepsilon}\|u\|_{M_1} \leqslant \inf_{k>0} \frac{1}{k}\left(1 + \int_G M[ku(x)]\,dx\right) \leqslant \|u\|_{M_1}.$$

From these inequalities and (10.14), it follows that

$$\frac{1}{(1+\varepsilon)^2}\|u\|_M \leqslant \inf_{k>0} \frac{1}{k}\left(1 + \int_G M[ku(x)]\,dx\right) \leqslant (1+\varepsilon)\|u\|_M.$$

Since ε is arbitrary, formula (10.11) is valid for the function $u(x)$. ★

§ 11. Compactness criteria

1. *Vallée Poussin's theorem.* We recall that a family $\mathfrak{N}$ of functions $\varphi(x)$ has equi-absolutely continuous integrals if for arbitrary $\varepsilon > 0$ an $h > 0$ can be found such that for all functions in the family $\mathfrak{N}$ we have

$$\int_{\mathscr{E}} |\varphi(x)|\,dx < \varepsilon$$

provided mes $\mathscr{E} < h$. A general criterion for the equi-absolute continuity of the integrals of a family of functions is given by the following theorem.

VALLÉE POUSSIN'S THEOREM (cf., e.g., NATANSON [1], p. 159, American edition). *Let* $\Phi(u)$ $(0 \leqslant u < \infty)$ *be a monotonically increasing function which satisfies the condition*

$$\lim_{n\to\infty} \frac{\Phi(u)}{u} = \infty.$$

Suppose, for the functions $\varphi(x)$ *of some family* $\mathfrak{N}$, *the integrals of the functions* $\Phi[|\varphi(x)|]$ *are uniformly bounded*:

$$\int_G \Phi[|\varphi(x)|]\,dx \leqslant A < \infty \quad (\varphi(x) \in \mathfrak{N}). \tag{11.1}$$

Then the family $\mathfrak{N}$ *has equi-absolutely continuous integrals.*

This straightforward theorem is proved in NATANSON [1] under somewhat more restrictive hypotheses.

Every N-function $M(u)$ satisfies the conditions of Vallée Poussin's theorem. Therefore, if the family of functions $\mathfrak{N}$ is contained in L_M and

$$\rho(u; M) = \int_G M[u(x)]\,dx \leqslant A \qquad (u(x) \in \mathfrak{N}), \tag{11.2}$$

then the functions $u(x)$ have equi-absolutely continuous integrals.

Now, let $M(u)$ and $M_1(u)$ be two N-functions for which

$$\lim_{u\to\infty} \frac{M(u)}{M_1(u)} = \infty.$$

Then it follows from (11.2) that the family of functions $\mathfrak{N}_1 = \{M_1[u(x)]\}$ has equi-absolutely continuous integrals on G. To prove this, it suffices to note that the function $\Phi(u) = M[M_1^{-1}(u)]$ satisfies the conditions of Vallée Poussin's theorem inasmuch as

$$\lim_{u\to\infty} \frac{\Phi(u)}{u} = \lim_{u\to\infty} \frac{M[M_1^{-1}(u)]}{u} = \lim_{u\to\infty} \frac{M(v)}{M_1(v)} = \infty.$$

2. *Steklov functions.* Let $u(x)$ be a function which is summable on G. The function

$$u_r(x) = \frac{1}{m_r} \int_{T_r(x)} u(t)\,dt \qquad (x \in G), \tag{11.3}$$

where $T_r(x)$ is the n-dimensional sphere with radius r and center at the point $x \in G$ and m_r is the volume of this sphere, is called a *Steklov function.* We set the function $u(t)$ in integral (11.3) equal to zero for $t \bar{\in} G$.

LEMMA 11.1. *Suppose given a family of functions $\mathfrak{N} \subset L_M^*$ with uniformly bounded norms $\|u\|_M \leqslant A$, $u(x) \in \mathfrak{N}$. Then the family $\mathfrak{N}_r$ of Steklov functions $u_r(x)$, $u(x) \in \mathfrak{N}$, is compact with respect to the uniform norm (in the space C of continuous functions on G).*

PROOF. Since

$$\int_G M\left[\frac{u(x)}{A}\right] dx \leqslant \int_G M\left[\frac{u(x)}{\|u\|_M}\right] dx \leqslant 1 \tag{11.4}$$

for $u(x) \in \mathfrak{N}$, we have, in virtue of Young's inequality, that

$$|u_r(x)| \leqslant \frac{1}{m_r} \int\limits_{T_r(x)} |u(t)|\,dt \leqslant \frac{A}{m_r} \int\limits_{G} \frac{|u(x)|}{A}\,dx \leqslant$$

$$\leqslant \frac{A}{m_r}(1 + N(1) \operatorname{mes} G).$$

Thus, the functions of the family $\mathfrak{N}_r$ are uniformly bounded.

It follows from Vallée Poussin's theorem, in virtue of (11.4), that the functions of the family $\mathfrak{N}$ have equi-absolutely continuous integrals, i.e. for given $\varepsilon > 0$ an $h > 0$ can be found such that

$$\int\limits_{\mathscr{E}} |u(x)|\,dx < \varepsilon \tag{11.5}$$

for all functions in the family, provided $\operatorname{mes} \mathscr{E} < h$ ($\mathscr{E} \subset G$).

We denote the set $(T_r(x) \cup T_r(y)) \setminus (T_r(x) \cap T_r(y))$ by $T_{x,y}$. Suppose that the volume of $T_{x,y}$ is less than h for $d(x, y) < \delta$. Then for all functions in the family $\mathfrak{N}_r$ we have, in virtue of (11.5), that

$$|u_r(x) - u_r(y)| \leqslant \frac{1}{m_r} \int\limits_{T_{x,y}} |u(t)|\,dt < \frac{\varepsilon}{m_r}$$

for $d(x, y) < \delta$, from which it follows that the functions of the family $\mathfrak{N}_r$ are equi-continuous. The assertion of the lemma now follows from Arzelà's theorem.

We shall use the following inequality in the sequel:

$$\|u_r\|_M \leqslant \|u\|_M. \tag{11.6}$$

To prove this inequality, we denote the sphere with radius r and center at the zero of the n-dimensional space by T_0. Then

$$\|u_r\|_M \leqslant \frac{1}{m_r} \sup_{\rho(v;\,N) \leqslant 1} \int\limits_{G} \int\limits_{T_r(x)} |u(t)v(x)|\,dt\,dx =$$

$$= \frac{1}{m_r} \sup_{\rho(v;\,N) \leqslant 1} \int\limits_{G} \int\limits_{T_0} |u(x + s)v(x)|\,ds\,dx.$$

Interchanging the order of integration in the last integral and

bringing the supremum sign under the integral sign, we obtain that

$$\|u_r\|_M \leqslant \frac{1}{m_r} \int_{T_0} \left[\sup_{\rho(v;\, N)\leqslant 1} \int_G |u(x+s)v(x)|\, dx \right] ds = \|u(x+s)\|_M,$$

and, since $u(t) = 0$ for $t \bar{\in} G$, we have

$$\|u(x+s)\|_M = \sup_{\rho(v;\, N)\leqslant 1} \int_G |u(x+s)v(x)|\, dx \leqslant$$

$$\leqslant \sup_{\rho(v;\, N)\leqslant 1} \int_G |u(t)v(t-s)|\, dt \leqslant \|u\|_M. \qquad (11.7)$$

This completes the proof of inequality (11.6).

3. *A. N. Kolmogorov's compactness criterion for the space* E_M.

THEOREM 11.1. *A family* $\mathfrak{N}$ *of functions of the space* E_M *is compact if, and only if, the following conditions are satisfied*:

a) $\|u\|_M \leqslant A$, $u(x) \in \mathfrak{N}$;

b) *for arbitrary* $\varepsilon > 0$, *a* $\delta > 0$ *can be found such that the condition* $r < \delta$ *implies that* $\|u - u_r\|_M < \varepsilon$ *for all functions of the family* $\mathfrak{N}$.

PROOF. The sufficiency of conditions a) and b) follows directly from Lemma 11.1 and Fréchet's theorem since a set of functions which is compact in C is also compact in every Orlicz space. [FRÉCHET'S THEOREM: *A set is compact if it can be approximated arbitrarily closely by a compact set.*]

Suppose $\mathfrak{N}$ is a compact family of functions in E_M. Then one can construct, for this family, an $(\varepsilon/3)$-net consisting of continuous functions $u^{(1)}(x)$, $u^{(2)}(x)$, $\cdots$, $u^{(n)}(x)$. The functions $u^{(i)}(x)$ $(i = 1, 2, \cdots, n)$ may or may not belong to the family $\mathfrak{N}$. We denote by c the norm of the characteristic function $\kappa(x; G)$ of the entire set G. Let $r > 0$ be a number such that for $d(x, t) < r$ we have

$$|u^{(i)}(x) - u^{(i)}(t)| < \varepsilon/(3c) \quad (i = 1, 2, \cdots, n).$$

Then

$$|u_r^{(i)}(x) - u^{(i)}(x)| \leqslant \frac{1}{m_r} \int_{T_r(x)} |u^{(i)}(t) - u^{(i)}(x)|\, dt \leqslant$$

$$\leqslant \frac{\varepsilon}{3c} \quad (i = 1, 2, \cdots, n)$$

and

$$\|u_r^{(i)} - u^{(i)}\|_M \leqslant \frac{\varepsilon}{3c} \|\kappa(x; G)\|_M = \frac{\varepsilon}{3}. \tag{11.8}$$

Suppose $u(x)$ is an arbitrary function in $\mathfrak{N}$. A function $u^{(i_0)}(x)$ can be found such that $\|u - u^{(i_0)}\|_M < \varepsilon/3$. Then in virtue of (11.6) and (11.8), we have that

$$\|u - u_r\|_M \leqslant \|u - u^{(i_0)}\|_M + \|u^{(i_0)} - u_r^{(i_0)}\|_M + \|u_r^{(i_0)} - u_r\|_M \leqslant$$

$$\leqslant 2\|u - u^{(i_0)}\|_M + \|u^{(i_0)} - u_r^{(i_0)}\|_M < \varepsilon.$$

The necessity of condition b) is thus proved. The necessity of condition a) is obvious. ★

In the case when the N-function $M(u)$ satisfies the Δ_2-condition, $E_M = L_M = L_M^*$. Therefore, in the case when the Δ_2-condition is satisfied, Theorem 11.1 yields a necessary and sufficient criterion for the compactness of a family of functions in L_M. Since, in this case, convergence in norm is equivalent to mean convergence, the following theorem is valid.

THEOREM 11.2. *Suppose the N-function $M(u)$ satisfies the Δ_2-condition. A necessary and sufficient condition that a family of functions $\mathfrak{N} \subset L_M = L_M^*$ be compact is that the following two conditions be satisfied*:

a) $\int_G M[u(x)]\,dx \leqslant A$, $u(x) \in \mathfrak{N}$;

b) *for arbitrary $\varepsilon > 0$, a $\delta > 0$ can be found such that*

$$\int_G M[u(x) - u_r(x)]\,dx < \varepsilon$$

for all functions in the family $\mathfrak{N}$, provided $r < \delta$.

4. *A second criterion for compactness.* We shall say that the family $\mathfrak{N}$ of functions $u(x) \in L_M^*$ have *equi-absolutely continuous norms* if for every $\varepsilon > 0$ a $\delta > 0$ can be found such that $\|u\kappa(x; \mathscr{E})\|_M < \varepsilon$ for all functions of the family $\mathfrak{N}$ provided mes $\mathscr{E} < \delta$. Clearly, in this case we have that $\mathfrak{N} \subset E_M$.

If $\mathfrak{N}$ is a compact set in E_M, then it can be shown, by means of usual reasoning, that it has equi-absolutely continuous norms.

LEMMA 11.2. *A necessary and sufficient condition that a sequence of functions $u_n(x) \in E_M$ $(n = 1, 2, \cdots)$, which converges in measure, converge in norm is that it have equi-absolutely continuous norms.*

PROOF. The necessity of the condition follows from the fact that a convergent sequence is compact. We shall prove the sufficiency of the condition of the lemma.

Suppose the sequence $u_n(x) \in E_M$ $(n = 1, 2, \cdots)$ converges in measure and has equi-absolutely continuous norms. Let $\varepsilon > 0$ be prescribed. We denote by $\mathscr{E}_{mn}$ the sets $G\{|u_n(x) - u_m(x)| > \eta\}$, where $\eta = \varepsilon/\{3 \text{ mes } GN^{-1}(1/\text{mes } G)\}$.

Let $\delta > 0$ be a number such that

$$\|u_n \kappa(x; \mathscr{E})\|_M < \frac{\varepsilon}{3} \qquad (n = 1, 2, \cdots)$$

provided mes $\mathscr{E} < \delta$.

Since the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ converges in measure, an n_0 can be found such that mes $\mathscr{E}_{mn} < \delta$ for $n, m > n_0$.

Then, for $n, m > n_0$, we have that

$$\begin{aligned} \|u_n - u_m\|_M &\leqslant \\ &\leqslant \|(u_n - u_m)\kappa(x; \mathscr{E}_{mn})\|_M + \|(u_n - u_m)\kappa(x; G/\mathscr{E}_{mn})\|_M \leqslant \\ &\leqslant \|u_m \kappa(x; \mathscr{E}_{mn})\|_M + \|u_m \kappa(x; \mathscr{E}_{mn})\|_M + \eta \|\kappa(x; G)\|_M < \varepsilon. \end{aligned}$$

This means that the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ converges in L_M^*. ★

THEOREM 11.3. *If the family $\mathfrak{N} \subset E_M$ has equi-absolutely continuous norms and is compact in the sense of convergence in measure, then the family $\mathfrak{N}$ is compact in L_M^*.*

PROOF. From each sequence of the family $\mathfrak{N}$ we can select a subsequence which converges in measure. In virtue of Lemma 11.2, this subsequence converges with respect to the norm in L_M^*. ★

Verification of the second condition of the theorem on compactness in the sense of convergence in measure usually reduces to the proof of the fact that the family $\mathfrak{N}$ is compact in some Orlicz space which is distinct from L_M^*.

5. *F. Riesz's criterion for compactness for the spaces E_M.* We shall give one more criterion for compactness of a family of functions in E_M.

THEOREM 11.4. *A family $\mathfrak{N}$ of functions in the space E_M is compact if, and only if, the following two conditions are satisfied*:

a) $\|u\|_M \leqslant A$, $u(x) \in \mathfrak{N}$;

b) *for arbitrary $\varepsilon > 0$, a $\delta > 0$ can be found such that $d(h, 0) < \delta$ implies that $\|u(x + h) - u(x)\|_M < \varepsilon$ for all $u(x) \in \mathfrak{N}$.*

PROOF. Suppose $u(x) \in \mathfrak{N}$ and that $u_r(x)$ is the corresponding Steklov function. Then

$$|u(x) - u_r(x)| \leqslant \frac{1}{m_r} \int_{T_r(x)} |u(x) - u(t)|\, dt,$$

from which it follows, for $v(x) \in L_N$, $\rho(v; N) \leqslant 1$, that

$$\int_G |u_r(x) - u(x)|\, v(x)\, dx \leqslant \frac{1}{m_r} \int_G \left[\int_{T_r(x)} |u(t) - u(x)|\, dt \right] v(x)\, dx.$$

Interchanging the order of integration and making a change in variables, we obtain that

$$\int_G |u(x) - u_r(x)|\, v(x)\, dx \leqslant$$

$$\leqslant \frac{1}{m_r} \int_{T_0} \left[\int_G |u(x + s) - u(x)|\, v(x)\, dx \right] ds \leqslant$$

$$\leqslant \frac{1}{m_r} \int_{T_0} \|u(x + s) - u(x)\|_M\, ds,$$

from which it follows that

$$\|u - u_r\|_M = \sup_{\rho(v; N) \leqslant 1} \left| \int_G [u(x) - u_r(x)]\, v(x)\, dx \right| \leqslant$$

$$\leqslant \frac{1}{m_r} \int_{T_0} \|u(x + s) - u(x)\|_M\, ds.$$

It follows from the last inequality that condition b) of Theorem 11.1 is satisfied provided condition b) of the present theorem is satisfied. Conditions a) of these theorems coincide.

This completes the proof of the sufficiency of the conditions of the theorem.

We shall now prove the necessity of these conditions. Suppose $\mathfrak{N}$ is a compact family of functions in E_M. Then, for this family, we can construct an $(\varepsilon/3)$-net consisting of continuous functions $u^{(1)}(x), u^{(2)}(x), \cdots, u^{(n)}(x)$. We denote by c the norm of the characteristic function of the entire set G in the space L_M^*.

Let $\delta > 0$ be a number such that

$$|u^{(i)}(x+h) - u^{(i)}(x)| < \frac{\varepsilon}{3c} \qquad (i = 1, 2, \cdots, n)$$

provided $d(h, 0) < \delta$. It is then clear that

$$\|u^{(i)}(x+h) - u^{(i)}(x)\|_M \leqslant \frac{\varepsilon}{3} \qquad (i = 1, 2, \cdots, n). \tag{11.9}$$

Let $u(x)$ be an arbitrary function in $\mathfrak{N}$. A $u^{(i_0)}(x)$ can be found such that $\|u - u^{(i_0)}\|_M < \varepsilon/3$. Then, in virtue of (11.9) and (11.7), we have that

$$\|u(x+h) - u(x)\|_M \leqslant \|u(x+h) - u^{(i_0)}(x+h)\|_M + \\ + \|u^{(i_0)}(x+h) - u^{(i_0)}(x)\|_M + \|u^{(i_0)}(x) - u(x)\|_M < \varepsilon.$$

The necessity of condition b) is thus proved. The necessity of condition a) is manifest. ★

§ 12. Existence of a basis

1. *Transition to the space of functions defined on a closed interval.* Below, we shall consider Orlicz spaces of functions defined on a finite segment. This will not disturb the generality of the results. This follows from the fact that the space $L_M^*(G)$ is linearly isometric to the space $L_M^*([0, \operatorname{mes} G])$, i.e. there exists a linear one-to-one correspondence between the elements of the spaces $L_M^*([0, \operatorname{mes} G])$ and $L_M^*(G)$, under which norms are preserved.

For the sake of simplicity of exposition, we shall show this here for the case when G is a bounded closed set situated in the plane with the cartesian coordinate system $(\mathscr{E}_1, \mathscr{E}_2)$.

We enclose the set G in a square B_0: $|\mathscr{E}_1| \leqslant b$, $|\mathscr{E}_2| \leqslant b$. We consider the sequence T_n of subdivisions of the square B_0 into 4^n

squares B_n^k $(k = 1, 2, \cdots, 4^n)$ by means of the straight lines

$$\mathscr{E}_1 = \frac{b}{2^{n-1}} i \text{ and } \mathscr{E}_2 = \frac{b}{2^{n-1}} j \quad (i, j = 0, \pm 1, \pm 2, \cdots, \pm 2^{n-1}).$$

In the transition from the subdivision T_n to the subdivision T_{n+1}, clearly, every square in the subdivision T_n subdivides into four equal squares in the subdivision T_{n+1} (Fig. 9).

We shall construct a mapping of the set of squares of all subdivisions onto the totality of certain segments — subsegments of the segment $I_0 = [0, \text{mes } G]$, in which the length of the segment I_n^k — the image of the square B_n^k — equals mes $(G \cap B_n^k)$. Suppose the entire segment I_0 corresponds to the square B_0. Let the segment $[\alpha, \beta] \subset I_0$ correspond to some square B_n^k in the subdivision T_n.

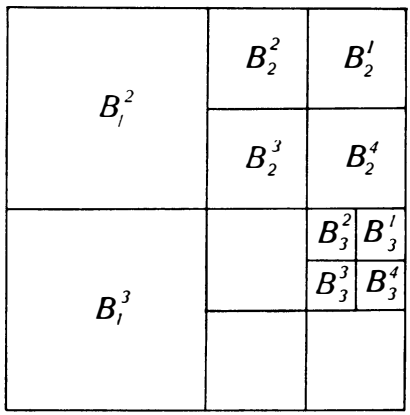

Fig. 9.

We enumerate the four squares in the subdivision T_{n+1} which comprise B_n^k as the quadrants are numbered in analytic geometry and we denote them by B^{I}, B^{II}, B^{III}, B^{IV}. The segment $[\alpha, \beta]$ is subdivided by means of the points $\gamma_1, \gamma_2, \gamma_3$, where $\alpha \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \beta$, into four parts such that

$$\gamma_1 - \alpha = \text{mes } (G \cap B^{\text{I}}), \quad \gamma_2 - \gamma_1 = \text{mes } (G \cap B^{\text{II}}),$$

$$\gamma_3 - \gamma_2 = \text{mes } (G \cap B^{\text{III}}), \quad \beta - \gamma_3 = \text{mes } (G \cap B^{\text{IV}}).$$

We correspond the segments $[\alpha, \gamma_1]$, $[\gamma_1, \gamma_2]$, $[\gamma_2, \gamma_3]$ and $[\gamma_2, \beta]$ with the squares B^{I}, B^{II}, B^{III} and B^{IV}, respectively.

To every set $P \subset G$ we assign the set $Q \subset I_0$ of points defined by the systems of contracting segments which correspond to systems of squares (i.e. subdivisions T_n), which contract to the points of the set P. Under this mapping, the measure of sets is preserved (this becomes clear if we note that the image of the totality of all sides of the squares of all subdivisions T_n has measure zero). If we denote by G_1 the set of points $x \in G$ any neighborhood of which contains a part of positive measure of the set G and which does not lie on the sides of the squares in the subdivision T_n, then $\operatorname{mes} G_1 = \operatorname{mes} G$, and the mapping σ, constructed above, of the set G onto the segment I_0 will be one-to-one on G_1. The image of the set G_1 will be denoted by Q_1.

Since the functions belonging to an Orlicz space are defined to within a set of measure zero, we can assume that all of them are equal to zero on $G \setminus G_1$. We correspond to each function $u(x) \in L_M^*(G)$ the function $\tilde{u}(t)$, $t \in I_0$, defined by

$$\tilde{u}(t) = \begin{cases} u(x) & \text{for } t = \sigma(x) \in Q_1, \\ 0 & \text{for } t \bar{\in} Q_1. \end{cases}$$

This correspondence is manifestly linear. Inasmuch as the measure of sets is preserved under the mapping σ, the functions in $L_M^*(G)$ are mapped, with preservation of norm, into functions in the space $L_M^*([0, \operatorname{mes} G])$.

It is also clear that the space $E_M(G)$ maps, under this mapping, into the space $E_M([0, \operatorname{mes} G])$.

2. *Haar functions.* The functions defined on the segment $[0, 1]$ by means of the following formulas are called *Haar functions*:

$$\chi_0^{(0)}(x) \equiv 1, \quad \chi_0^{(0)}(x) = \begin{cases} 1 \text{ for } 0 \leqslant x < \frac{1}{2}, \\ 0 \text{ for } x = \frac{1}{2}, \\ -1 \text{ for } \frac{1}{2} < x \leqslant 1 \end{cases}$$

and, further, for $n = 1, 2, \cdots$; $k = 1, 2, \cdots, 2^n$,

$$\chi_n^{(k)}(x) = \begin{cases} \sqrt{2^n} \text{ for } \dfrac{2k-2}{2^{n+1}} \leqslant x < \dfrac{2k-1}{2^{n+1}}, \\ -\sqrt{2^n} \text{ for } \dfrac{2k-1}{2^{n+1}} < x \leqslant \dfrac{2k}{2^{n+1}}, \\ 0 \quad \text{for the remaining values of } x. \end{cases}$$

We write the Haar functions $\chi_n^{(k)}(x)$ as a sequence in the order of increasing n, and for given n in the order of increasing k. The sequence thus obtained is denoted by $\varphi_i(x)$ $(i = 1, 2, \cdots)$. This sequence of functions, as is easily seen, is orthogonal:

$$\int_0^1 \varphi_i(x)\varphi_j(x)\,dx = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Since all Haar functions are bounded, the Fourier coefficients

$$c_i = \int_0^1 u(x)\varphi_i(x)\,dx \quad (i = 1, 2, \cdots) \tag{12.1}$$

are defined for every summable function $u(x)$.

We denote the operators $\boldsymbol{S}_m$ by means of the equality

$$\boldsymbol{S}_m u(x) = \sum_{i=1}^{m} c_i\varphi_i(x) \quad (m = 1, 2, \cdots).$$

We denote the points of discontinuity of the functions $\varphi_1(x), \varphi_2(x), \cdots, \varphi_m(x)$ to which the points 0 and 1 are added by $a_1, a_2, \cdots, a_p$ arranged in increasing order.

LEMMA 12.1. *For $a_s < x < a_{s+1}$, the value of the piecewise constant function $\boldsymbol{S}_m u(x)$ is defined by the equality*

$$\boldsymbol{S}_m u(x) = \frac{1}{a_{s+1} - a_s} \int_{a_s}^{a_{s+1}} u(x)\,dx.$$

PROOF. It follows directly from the definition of Haar functions that the function

$$F_m(x, y) = \sum_{i=1}^{m} \varphi_i(x)\varphi_i(y)$$

assumes, for $a_s < x < a_{s+1}$, the following values:

$$F_m(x, y) = \begin{cases} \dfrac{1}{a_{s+1} - a_s} & \text{if } a_s < y < a_{s+1}, \\ 0 & \text{if } y < a_s \text{ or } a_{s+1} < y \end{cases}$$

(this is verified most easily by induction on m). Therefore, for

$a_s < x < a_{s+1}$, we have that

$$\boldsymbol{S}_m u(x) = \sum_{i=1}^{m} \varphi_i(x) \int_0^1 u(y)\varphi_i(y)\,dy = \int_0^1 u(y) F_m(x, y)\,dy =$$

$$= \frac{1}{a_{s+1} - a_s} \int_{a_s}^{a_{s+1}} u(y)\,dy. \ \bigstar$$

3. *Basis in E_M.* Let $M(u)$ and $N(v)$ be mutually complementary N-functions.

Suppose $u(x)$ belongs to the space $L_M([0, 1])$. From the Jensen integral inequality and Lemma 12.1, it follows that

$$M[\boldsymbol{S}_m u(x)] = M\left[\frac{1}{a_{s+1} - a_s} \int_{a_s}^{a_{s+1}} u(x)\,dx\right] \leqslant$$

$$\leqslant \frac{1}{a_{s+1} - a_s} \int_{a_s}^{a_{s+1}} M[u(x)]\,dx$$

for every m and $a_s < x < a_{s+1}$, which implies that

$$\int_{a_s}^{a_{s+1}} M[\boldsymbol{S}_m u(x)]\,dx \leqslant \int_{a_s}^{a_{s+1}} M[u(x)]\,dx$$

and

$$\int_0^1 M[\boldsymbol{S}_m u(x)]\,dx \leqslant \int_0^1 M[u(x)]\,dx. \tag{12.2}$$

If $\|u\|_M \leqslant 1$, then, in virtue of (9.13), we have that

$$\int_0^1 M[u(x)]\,dx \leqslant 1.$$

It therefore follows from inequalities (12.2) and (9.12) that

$$\|\boldsymbol{S}_m u\|_M \leqslant 1 + \int_0^1 M[\boldsymbol{S}_m u(x)]\,dx \leqslant 1 + \int_0^1 M[u(x)]\,dx \leqslant 2$$

for $\|u\|_M \leqslant 1$. Consequently,

$$\|S_m\| = \sup_{\|u\|_M \leqslant 1} \|S_m u\|_M \leqslant 2 \quad (m = 1, 2, \cdots).$$

Thus, the norms of the linear operators S_m, acting in the space $L_M^*([0, 1])$, are uniformly bounded.

Now suppose the function $u(x)$ is continuous. In virtue of Lemma 12.1, the sequence $S_m u(x)$ converges to $u(x)$ uniformly on the segment $[0, 1]$ if one discards from it the denumerable number of points of discontinuity of all Haar functions. Therefore the functions $S_m u(x)$ also converge in norm to $u(x)$ in any Orlicz space.

So, the sequence of operators S_m $(m = 1, 2, \cdots)$ has uniformly bounded norms and converges strongly to the identity operator on a set of continuous functions which is dense in $E_M([0, 1])$. In virtue of the known Banach-Steinhaus theorem, the operators S_m converge to the identity operator on all of $E_M([0, 1])$. This means that the series

$$\sum_{i=1}^{\infty} c_i \varphi_i(x), \tag{12.3}$$

where the c_i are defined by equality (12.1), converges in L_M^* to $u(x)$ for arbitrary function $u(x) \in E_M([0, 1])$.

Thus the Haar functions $\varphi_i(x)$ $(i = 1, 2, \cdots)$ form a basis in $E_M([0, 1])$.

It is clear that the functions $\varphi_i(x/\operatorname{mes} G)$ $(i = 1, 2, \cdots)$, defined on the segment $[0, \operatorname{mes} G]$, form a basis in $E_M([0, \operatorname{mes} G])$. This fact and the line of reasoning followed in the first subsection of this section lead to the following assertion.

THEOREM 12.1. *A basis exists in the space* $E_M(G)$.

It follows from this theorem that a basis exists for the entire space L_M^* if the N-function $M(u)$ satisfies the Δ_2-condition inasmuch as $E_M = L_M^*$ in this case.

The Fourier coefficients (12.1) are defined for all functions $u(x) \in L_M^*([0, 1])$ and not only for functions in $E_M([0, 1])$. For those functions $u(x) \in L_M^*([0, 1])$ for which series (12.3) converges, one could have defined an operator P by setting $Pu(x) = \sum_{i=1}^{\infty} c_i \varphi_i(x)$. The operator P would be an operator of projection on E_M.

It turns out that the set $L_M^* \setminus E_M$ contains no functions for which series (12.3) converges in L_M^*. In fact, suppose series (12.3) converges

for some function $u(x)$. We then set

$$g(x) = u(x) - \sum_{i=1}^{\infty} c_i \varphi_i(x).$$

All Fourier coefficients (12.1) for the function $g(x)$ are equal to zero. In virtue of Theorem 12.1, every function $u(x) \in E_N([0, 1])$ can be represented in the form of a series, $v(x) = \sum_{i=1}^{\infty} d_i \varphi_i(x)$, which converges in $L_N^*([0, 1])$. This enables us to compute the norm of the function $g(x)$: in virtue of (10.4), we have that

$$\|g\|_M = \sup_{\substack{\rho(v;\, N) \leqslant 1 \\ v \in E_N}} \left| \int_0^1 g(x) v(x)\, dx \right| = \sup_{\substack{\rho(v;\, N) \leqslant 1 \\ v \in E_N}} \left| \sum_{i=1}^{\infty} d_i \int_0^1 g(x) \varphi_i(x)\, dx \right| = 0.$$

This means that $g(x) = 0$ and $u(x) \in E_M([0, 1])$.

4. *Further remarks on the conditions for separability.* It was shown in the preceding section that the space L_M^* is not separable if the N-function $M(u)$ does not satisfy the Δ_2-condition. This same fact can be proved by effectively demonstrating a continuous totality of functions in the space L_M^* the mutual distances between which is greater than some fixed number. The reasoning of the first subsection permits us to limit ourselves to the construction of such a set in the space $L_M^*([0, 1])$.

Suppose the N-function $M(u)$ does not satisfy the Δ_2-condition; then a sequence of positive numbers $u_1 < u_2 < \cdots < u_n < \cdots \to \infty$ can be found such that $M(2u) > 2^n M(u_n)$ $(n = 1, 2, \cdots)$.

We construct on $[0, 1)$ a sequence of disjoint segments δ_n arranged in order of increasing indices — from left to right — whose lengths are determined by the equalities

$$\operatorname{mes} \delta_n = \frac{M(u_1)}{2^n M(u_n)} \qquad (n = 1, 2, \cdots).$$

Such a construction is possible inasmuch as $\sum_{n=1}^{\infty} \operatorname{mes} \delta_n < 1$. Suppose, furthermore, that unity is a limit point of $\bigcup_{n=1}^{\infty} \delta_n$.

We define a function $u(x)$ on $[0, 1]$ by setting

$$u(x) = \begin{cases} 2u_n & \text{for } x \in \delta_n \quad (n = 1, 2, \cdots), \\ 0 & \text{for } x \bar{\in} \bigcup_{n=1}^{\infty} \delta_n. \end{cases}$$

The function $u(x)$ belongs to L_M^* since $\frac{1}{2}u(x) \in L_M$:

$$\int_0^1 M[\tfrac{1}{2}u(x)]\,dx = \sum_{n=1}^{\infty} \int_{\delta_n} M[\tfrac{1}{2}u(x)]\,dx = \sum_{n=1}^{\infty} M(u_n) \text{ mes } \delta_n < \infty.$$

Suppose, for all $0 < \alpha \leqslant 1$, that the functions $\varphi_\alpha(x)$ are defined by the equality

$$\varphi_\alpha(x) = \begin{cases} u(x + 1 - \alpha) & \text{for } 0 \leqslant x \leqslant \alpha, \\ u(x - \alpha) & \text{for } \alpha < x \leqslant 1. \end{cases}$$

All these functions belong to L_M^*.

We consider two functions $\varphi_\alpha(x)$ and $\varphi_\beta(x)$, where $\alpha < \beta$.

By construction, the function $\varphi_\beta(x)$ is bounded on the segment $[0, \alpha]$. Suppose $|\varphi_\beta(x)| = \varphi_\beta(x) \leqslant A$ $(0 \leqslant x \leqslant \alpha)$. Then a positive number $\eta < \alpha$ can be found such that

$$\|\varphi_\beta \kappa_\eta\|_M < \tfrac{1}{2}, \tag{12.4}$$

where $\kappa_\eta(x)$ is the characteristic function of the segment $[\alpha - \eta, \alpha]$. In fact, in virtue of formula (9.11), we have

$$\|\varphi_\beta \kappa_\eta\|_M \leqslant A\|\kappa_\eta\|_M = A\eta N^{-1}\left(\frac{1}{\eta}\right) < \tfrac{1}{2}$$

for the norm of the characteristic function, for sufficiently small η.

We shall estimate the norm of the function $\varphi_\alpha(x)\kappa_\eta(x)$ from below. Clearly, this norm is equal to the norm of the function $u(x)\hat{\kappa}_\eta(x)$, where $\hat{\kappa}_\eta(x)$ is the characteristic function of the segment $[1 - \eta, 1]$. We consider the sets

$$F_n = [1 - \eta, 1] \cap \left(\bigcup_{i=1}^{n} \delta_i\right) \quad (n = 1, 2, \cdots).$$

The characteristic functions of these sets will be denoted by $\kappa_n(x)$ $(n = 1, 2, \cdots)$. The function $u(x)$ is bounded on each of the sets F_n so that the functions $v_n(x) = p[u(x)\kappa_n(x)]$ $(n = 1, 2, \cdots)$

are also bounded and, a fortiori, belong to the class L_N. In virtue of (2.7), we have that

$$\int_0^1 u(x)v_n(x)\,dx = \int_{F_n} M[u(x)]\,dx + \int_0^1 N[v_n(x)]\,dx.$$

We shall show that

$$\lim_{n\to\infty} \int_0^1 N[v_n(x)]\,dx > 1. \tag{12.5}$$

In fact, if the inequality

$$\int_0^1 N[v_n(x)]\,dx \leqslant 1$$

were true for all n $(n = 1, 2, \cdots)$, then the function $u(x)\hat{\kappa}_\eta(x)$ would belong to L_M inasmuch as

$$\int_0^1 M[u(x)\hat{\kappa}_\eta(x)]\,dx \leqslant \sup_{n=1,2,\cdots} \left\{ \int_{F_n} M[u(x)]\,dx + \right.$$

$$\left. + \int_{F_n} N[v_n(x)]\,dx \right\} = \sup_{n=1,2,\cdots} \int_0^1 u(x)v_n(x)\,dx \leqslant$$

$$\leqslant \sup_{\rho(v;\,N)\leqslant 1} \int_0^1 u(x)v(x)\,dx = \|u\|_M < \infty,$$

which is a contradiction because

$$\int_0^1 M[u(x)\hat{\kappa}_\eta(x)]\,dx = \int_0^1 M[u(x)]\,dx - \int_0^{1-\eta} M[u(x)]\,dx =$$

$$= \sum_{n=1}^{\infty} \frac{M(2u_n)M(u_1)}{2^n M(u_n)} - \sup_{x\leqslant 1-\eta} M[u(x)] = \infty.$$

In virtue of (12.5), an n_0 can be found such that

$$\int_0^1 N[v_{n_0}(x)]\,dx = \rho > 1.$$

Then, in virtue of (1.17), we have that

$$\int_0^1 N\left[\frac{v_{n_0}(x)}{\rho}\right] dx \leqslant 1.$$

Thus, the following estimate is valid for $\|u\hat{\kappa}_\eta\|_M$:

$$\|u\hat{\kappa}_\eta\|_M = \sup_{\rho(v;\,N)\leqslant 1} \int_0^1 u(x)\hat{\kappa}_\eta(x)v(x)\,dx \geqslant \int_0^1 u(x)\hat{\kappa}_\eta(x)\,\frac{v_{n_0}(x)}{\rho}\,dx =$$

$$= \frac{1}{\rho}\left\{\int_{F_{n_0}} M[u(x)]\,dx + \int_0^1 N[v_{n_0}(x)]\,dx\right\} > 1. \qquad (12.6)$$

From this inequality and (12.4), it follows that

$$\|\varphi_\alpha - \varphi_\beta\| \geqslant \|\varphi_\alpha\kappa_\eta\|_M - \|\varphi_\beta\kappa_\eta\|_M = \|u\hat{\kappa}_\eta\|_M - \|\varphi_\beta\kappa_\eta\|_M >$$

$$> 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

Thus, the functions $\varphi_\alpha(x)$ $(0 < \alpha \leqslant 1)$ are at a distance greater than $\frac{1}{2}$ from one another.

§ 13. Spaces determined by distinct N-functions

1. *Comparison of spaces.* Generally speaking, distinct N-functions determine distinct Orlicz spaces. For example, the spaces L^α determined by the N-functions $M(u) = |u|^\alpha/\alpha$ are distinct for distinct $\alpha > 1$.

THEOREM 13.1. *Let $M_1(u)$ and $M_2(u)$ be two N-functions. A necessary and sufficient condition that L_{M_1} be $\subset L_{M_2}$ is that the relation $M_2(u) \prec M_1(u)$ be satisfied, i.e. that there exist constants u_0, $k > 0$ such that*

$$M_2(u) \leqslant M_1(ku) \quad (u \geqslant u_0). \qquad (13.1)$$

PROOF. Let us assume that condition (13.1) is not satisfied. Then an indefinitely increasing monotonic sequence of numbers u_n $(n = 1, 2, \cdots)$ can be found such that

$$M_2(u_n) > M_1(2^n n u_n) \quad (n = 1, 2, \cdots). \tag{13.2}$$

In virtue of (1.17), $M_1(nu_n)/(nu_n) \leqslant M_1(2^n nu_n)/(2^n nu_n)$, from which it follows that $M_1(2^n nu_n) \geqslant 2^n M_1(nu_n)$. Combining the last inequality with (13.2), we obtain that

$$M_2(u_n) > 2^n M_1(nu_n). \tag{13.3}$$

Suppose $G_1, G_2, \cdots$ are disjoint subsets of the set G for which

$$\operatorname{mes} G_n = \frac{M_1(u_1) \operatorname{mes} G}{2^n M_1(nu_n)} \quad (n = 1, 2, \cdots).$$

Such sets can be constructed inasmuch as $\sum_{n=1}^{\infty} \operatorname{mes} G_n < \operatorname{mes} G$.

Now we consider the function $u(x)$ defined by the equality

$$u(x) = \begin{cases} nu_n & \text{for } x \in G_n \quad (n = 1, 2, \cdots), \\ 0 & \text{for } x \bar{\in} \bigcup_{n=1}^{\infty} G_n. \end{cases}$$

This function belongs to the space $L^*_{M_1}$ inasmuch as

$$\int_G M_1[u(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M_1[u(x)]\,dx = \sum_{n=1}^{\infty} M_1(nu_n) \operatorname{mes} G_n =$$

$$= M(u_1) \operatorname{mes} G \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

This function does not, however, belong to the space $L^*_{M_2}$ since, for all $\lambda \geqslant 1$, the functions $(1/\lambda)u(x)$ do not belong to L_{M_2}. In fact, suppose m is an integer greater than λ. Then, in virtue of (13.3), we have that

$$\int_G M_2\left[\frac{u(x)}{\lambda}\right] dx = \sum_{n=1}^{\infty} M_2\left(\frac{nu_n}{\lambda}\right) \operatorname{mes} G_n \geqslant \sum_{n=m}^{\infty} M_2(u_n) \operatorname{mes} G_n \geqslant$$

$$\geqslant \sum_{n=m}^{\infty} 2^n M_1(nu_n) \operatorname{mes} G_n = \infty.$$

This proves the necessity of condition (13.1). We shall now prove the sufficiency of this condition.

Suppose condition (13.1) is satisfied. Then the function $u(x)$ belongs to the space $L^*_{M_1}$. This means that for some $\mu > 0$, $\mu u(x) \in L_{M_1}$, i.e. that

$$\int_G M_1[\mu u(x)]\,dx < \infty.$$

We denote the set $G\{|u(x)| < ku_0/\mu\}$ by G_0. Then, in virtue of (13.1), we have that

$$\int_G M_2\left[\frac{\mu u(x)}{k}\right]dx = \int_{G_0} M_2\left[\frac{\mu u(x)}{k}\right]dx + \int_{G \setminus G_0} M_2\left[\frac{\mu u(x)}{k}\right]dx \leqslant$$

$$\leqslant M_2(u_0)\ \text{mes}\ G_0 + \int_G M_1[\mu u(x)]\,dx < \infty.$$

This means that $(\mu/k)u(x) \in L_{M_2}$ from which it follows that $u(x) \in L^*_{M_2}$. ★

Let us recall that the N-functions $M_1(u)$ and $M_2(u)$ are said to be equivalent, $M_1(u) \sim M_2(u)$, if $M_1(u) \prec M_2(u)$ and $M_2(u) \prec M_1(u)$, i.e. if there exist positive constants k_1, k_2 and u_0 such that

$$M_1(k_1 u) \leqslant M_2(u) \leqslant M_1(k_2 u) \quad (u \geqslant u_0). \tag{13.4}$$

Theorem 13.1 implies the next theorem.

THEOREM 13.2. *The spaces $L^*_{M_1}$ and $L^*_{M_2}$ consist of the same functions if, and only if, the N-functions $M_1(u)$ and $M_2(u)$ are equivalent.*

This theorem is the basis for the introduction of the concept of equivalent N-functions.

2. *An inequality for norms.*

THEOREM 13.3. *If $L^*_{M_1} \subset L^*_{M_2}$, then there exists a constant $q > 0$ such that*

$$\|u\|_{M_2} \leqslant q\|u\|_{M_1}, \quad u(x) \in L^*_{M_1}. \tag{13.5}$$

PROOF. In virtue of Theorems 13.1 and 3.1, $L^*_{N_2} \subset L^*_{N_1}$ in which connection positive numbers k and v_0 can be found such that

$N_1(v/k) \leqslant N_2(v)$, $v \geqslant v_0$. Then, for all v, we have that

$$N_1\left(\frac{v}{k}\right) \leqslant N_1\left(\frac{v_0}{k}\right) + N_2(v). \tag{13.6}$$

Now suppose $v(x) \in L_N$ and $\rho(v; N_2) \leqslant 1$. Then, in virtue of (13.6), we have that

$$\rho\left(\frac{v}{k}; N_1\right) = \int_G N_1\left[\frac{v(x)}{k}\right] dx \leqslant N_1\left(\frac{v_0}{k}\right) \operatorname{mes} G +$$

$$+ \int_G N_2[v(x)]\, dx \leqslant N_1\left(\frac{v_0}{k}\right) \operatorname{mes} G + 1 = a.$$

We set $q = ak$. Then, in virtue of the last inequality and (1.17), we have that

$$\rho\left(\frac{v}{q}; N_1\right) = \int_G N_1\left[\frac{v(x)}{ak}\right] dx \leqslant \frac{1}{a} \int_G N_1\left[\frac{v(x)}{k}\right] dx \leqslant 1.$$

Relation (13.5) follows from the inequalities

$$\|u\|_{M_2} = \sup_{\rho(v;\, N_2) \leqslant 1} \left| \int_G u(x) v(x)\, dx \right| = q \sup_{\rho(v;\, N_2) \leqslant 1} \left| \int_G u(x) \frac{v(x)}{q} dx \right| \leqslant$$

$$\leqslant q \sup_{\rho(v;\, N) \leqslant 1} \left| \int_G u(x) w(x)\, dx \right| = q \|u\|_{M_1}. \bigstar$$

It is easily seen that inequality (13.5) is satisfied with the constant $q = 1$: $\|u\|_{M_2} \leqslant \|u\|_{M_1}$, $u(x) \in L^*_{M_1}$, if the inequality $M_2(u) \leqslant M_1(u)$ is satisfied for all u.

From Theorem 13.3, it follows, in particular, that the norms generated by equivalent N-functions $M_1(u)$ and $M_2(u)$ are equivalent:

$$q_1 \|u\|_{M_1} \leqslant \|u\|_{M_2} \leqslant q_2 \|u\|_{M_1}. \tag{13.7}$$

In the solution of many problems, one can therefore choose from the class of equivalent N-functions that one which, in virtue of any considerations, is the most convenient.

Let $M_\alpha(u) = M(\alpha u)$, $\alpha > 0$. Clearly $M_\alpha(u) \sim M(u)$. The equality

$$\|u\|_{M_\alpha} = \alpha \|u\|_M \tag{13.8}$$

holds, for the proof of which it suffices to note that $N_\alpha(v) = N\left(\frac{v}{\alpha}\right)$ and

$$\|u\|_{M_\alpha} = \sup_{\rho(v;\, N_\alpha)\leqslant 1} \left| \int_G u(x)v(x)\,dx \right| = \sup_{\rho(v/\alpha;\, N)\leqslant 1} \left| \int_G u(x)v(x)\,dx \right| =$$

$$= \alpha \sup_{\rho(v/\alpha;\, N)\leqslant 1} \left| \int_G u(x)\frac{v(x)}{\alpha}\,dx \right| = \alpha \|u\|_M.$$

3. *Concerning a criterion for convergence in norm.* We shall say that the N-function $M_1(u)$ *increases essentially more rapidly* than the N-function $M(u)$ if, for arbitrary $\lambda > 0$,

$$\lim_{u\to\infty} \frac{M(\lambda u)}{M_1(u)} = 0. \tag{13.9}$$

For example, the function $|u|^\alpha$ increases essentially more rapidly than the functions $|u|^\beta$ if $\alpha > \beta$. It is also easily seen that the composition $M_1(u) = M[Q(u)]$ of two N-functions $M(u)$ and $Q(u)$ increases essentially more rapidly than the N-function $M(u)$.

It is not difficult to see that $M_1(u)$ increases essentially more rapidly than $M(u)$ if, and only if, for every $\varepsilon > 0$, $M(u) < M_1(\varepsilon u)$ for large values of the argument.

LEMMA 13.1. *If $M_1(u)$ increases essentially more rapidly than $M(u)$, then $N(v)$ increases essentially more rapidly than $N_1(v)$, where $N(v)$ and $N_1(v)$ are functions complementary to the functions $M(u)$ and $M_1(u)$, respectively.*

PROOF. Let an arbitrarily small $\varepsilon > 0$ and any μ be prescribed. In virtue of (13.9), the inequality $M(u/\varepsilon) \leqslant M_1(u/\mu)$ is satisfied for large values of u.

In virtue of Theorem 2.1 and (2.5), the inequality $N_1(\mu v) \leqslant N(\varepsilon v)$ is satisfied for the complementary functions for large values of v, from which it follows, in virtue of (1.17), that $N_1(\mu v) \leqslant \varepsilon N(v)$. This means that

$$\lim_{v\to\infty} \frac{N_1(\mu v)}{N(v)} = 0. \bigstar \tag{13.10}$$

If $M_1(u)$ increases essentially more rapidly than $M(u)$, then the inclusion

$$L^*_{M_1} \subset E_M \tag{13.11}$$

holds.

Since E_M is the maximal linear subset of the class L_M and $L^*_{M_1}$ is the linear hull of the class L_{M_1}, it will suffice to show that, for arbitrary λ, the function $\lambda u(x)$ belongs to L_M, if $u(x) \in L_{M_1}$. The last assertion follows from the fact that, in virtue of (13.9), $M(\lambda u) \leqslant M_1(u)$ for large values of u.

LEMMA 13.2. *Suppose the N-function $M_1(u)$ increases essentially more rapidly than the N-function $M(u)$. Suppose also that the family of functions $\mathfrak{N}$ is uniformly bounded in the space $L^*_{M_1}$: $\|u\|_{M_1} \leqslant a$, $u(x) \in \mathfrak{N}$. Then the family has equi-absolutely continuous norms in L^*_M.*

PROOF. Let $\varepsilon > 0$ be an arbitrarily small prescribed number. We set $\mu = 2a/\varepsilon$. In virtue of (13.10) and the de la Vallée Poussin theorem (see § 11, subsection 1), the functions $N_1[\mu v(x)]$, where $\rho(v; N) \leqslant 1$, have equi-absolutely continuous integrals, i.e. a $\delta > 0$ can be found such that for all functions $v(x) \in L_N$, which satisfy the condition $\rho(v; N) \leqslant 1$, we have that

$$\int_{\mathscr{E}} N_1[\mu v(x)]\, dx < \frac{\varepsilon}{2},$$

provided mes $\mathscr{E} < \delta$ ($\mathscr{E} \subset G$).

Let $u(x) \in \mathfrak{N}$ and $v(x) \in L_N$, $\rho(v; N) \leqslant 1$. It then follows from Young's inequality that

$$\left| \int_{\mathscr{E}} u(x)v(x)\, dx \right| \leqslant \int_{\mathscr{E}} M_1\left[\frac{u(x)}{\mu}\right] dx + \int_{\mathscr{E}} N_1[\mu v(x)]\, dx \leqslant$$

$$\leqslant \left\| \frac{u}{\mu} \right\|_{M_1} + \frac{\varepsilon}{2} < \varepsilon$$

for mes $\mathscr{E} < \delta$, which implies that

$$\|u\kappa(x; \mathscr{E})\|_M = \sup_{\rho(v; N) \leqslant 1} \left| \int_{\mathscr{E}} u(x)v(x)\, dx \right| \leqslant \varepsilon$$

for all functions $u(x) \in \mathfrak{N}$ provided mes $\mathscr{E} < \delta$ ($\mathscr{E} \subset G$). ★

The converse assertion also holds.

LEMMA 13.3. *Let $M(u) \prec M_1(u)$. Suppose every set of functions bounded in $L^*_{M_1}$ has equi-absolutely continuous norms in L^*_M. Then the N-function $M_1(u)$ increases essentially more rapidly than $M(u)$.*

PROOF. Let us assume that $M_1(u)$ does not increase essentially more rapidly than $M(u)$. Then, in virtue of Lemma 13.1, the N-function $N(v)$, complementary to the function $M(u)$, does not increase essentially more rapidly than the complementary function $N_1(v)$ to $M_1(u)$. This means that an $\varepsilon_0 > 0$ and a sequence of numbers $v_n \to \infty$ can be found such that $N_1(v_n) > N(\varepsilon_0 v_n)$ $(n = 1, 2, \cdots)$.

It follows from these inequalities that the inequalities

$$N^{-1}(w_n) > \varepsilon_0 N_1^{-1}(w_n) \quad (n = 1, 2, \cdots) \tag{13.12}$$

are valid for the numbers $w_n = N_1(v_n)$.

We shall denote by G_n $(n = 1, 2, \cdots)$ those subsets of G for which mes $G_n = 1/w_n$. Let

$$u_n(x) = \begin{cases} \dfrac{w_n}{N_1^{-1}(w_n)} & \text{if } x \in G_n, \\ 0 & \text{if } x \bar{\in} G_n. \end{cases}$$

Clearly, $\|u_n\|_{M_1} = 1$. Since mes $G_n \to 0$, the relation $\lim\limits_{n\to\infty} \|u_n\|_M = 0$ must be satisfied in virtue of the condition of the lemma. This contradicts the following inequality, which is implied by (13.12):

$$\|u_n\|_M = \frac{w_n}{N_1^{-1}(w_n)} \|\kappa(x; G_n)\|_M = \frac{N^{-1}(w_n)}{N_1^{-1}(w_n)} > \varepsilon_0. \bigstar$$

THEOREM 13.4. *Suppose the N-function $M_1(u)$ increases essentially more rapidly than the function $M(u)$. Suppose also that the sequence $u_n(x) \in L_{M_1}$ $(n = 1, 2, \cdots)$ is mean convergent to zero:*

$$\lim_{n\to\infty} \int_G M_1[u_n(x)]\, dx = 0. \tag{13.13}$$

Then this sequence converges to zero with respect to the norm of the space L^*_M: $\lim\limits_{n\to\infty} \|u_n\|_M = 0$.

PROOF. It follows from condition (13.13) that the norms $\|u_n\|_{M_1}$ $(n = 1, 2, \cdots)$ are uniformly bounded. In virtue of Lemma (13.2),

the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ has equi-absolutely continuous norms in L^*_M. Since, in virtue of the same condition (13.13), the sequence $u_n(x)$ $(n = 1, 2, \cdots)$ converges in measure to zero, it converges, in virtue of Lemma 11.2, to zero with respect to the norm in L^*_M. ★

4. *The product of functions in Orlicz spaces.* Let $u(x) \in L^*_{M_1}$, $w(x) \in L^*_\Phi$. Generally speaking, the product $u(x)w(x)$ is not even a summable function. However, if the N-functions $M_1(u)$ and $\Phi(u)$ are interrelated in a definite way, then the product $u(x)w(x)$ may turn out to be not only summable but it may also belong to a third Orlicz space $L^*_{M_2}$, where $M_2(u)$ is defined by the functions $M_1(u)$ and $\Phi(u)$. In the present subsection, we shall investigate a number of problems which arise in this connection.

As the first example, we consider the case when $L^*_{M_1} = L^{\alpha_1}$, $L^*_\Phi = L^{\alpha_2}$. A necessary condition that the product $u(x)w(x)$ of the functions $u(x) \in L^{\alpha_1}$ and $w(x) \in L^{\alpha_1}$ be a summable function is, obviously, that the inequality $1/\alpha_1 + 1/\alpha_2 \leqslant 1$ be satisfied. In this connection, $u(x)w(x)$ belongs to all the L^γ, where $1 \leqslant \gamma \leqslant \alpha_1\alpha_2/(\alpha_1 + \alpha_2)$. The verification of these evident facts is left to the reader.

We proceed to the general case.

LEMMA 13.4. *Let $u(x)$ be a function. Suppose $u(x)w(x) \in E$ for all functions $w(x) \in L^*_\Phi$, where E is an Orlicz space $L^*_{M_2}$ or the space L of summable functions. Then there exists a constant $k > 0$ such that $\|uw\|_E \leqslant k\|w\|_\Phi$.*

PROOF. The formula

$$Aw(x) = u(x)w(x), \quad w(x) \in L^*_\Phi$$

defines an additive, homogeneous operator which acts from L^*_Φ into E. We shall show that this operator is closed. [We recall that the operator A is said to be *closed* if $\|w_n - w_0\|_\Phi \to 0$ and $\|Aw_n - v\|_E \to 0$ imply that $v = Aw_0$.] In fact, let $\|w_n - w_0\|_\Phi \to 0$ and $\|uw_n - v\|_E \to 0$. Then the functions $w_n(x)$ converge in measure to $w_0(x)$ in virtue of which the functions $u(x)w_n(x)$ converge in measure to $u(x)w_0(x)$. On the other hand, the functions $u(x)w_n(x)$ converge in measure to $v(x)$. It thus follows from this that $v(x) = u(x)w_0(x)$ almost everywhere. Since the closed operator A is defined on the entire Banach space L^*_Φ, it is continuous. [Every

closed, additive, homogeneous operator A defined on a complete metric space is continuous (see, e.g., HILLE [1]).] ★

LEMMA 13.5. *Let* $u(x)w(x) \in E$ *for all functions* $u(x) \in L^*_{M_1}$, $w(x) \in L^*_{\Phi}$, *where* E *is an Orlicz space* $L^*_{M_2}$ *or the space* L. *Then there exists a constant* $k > 0$ *such that*

$$\|uw\|_E \leqslant k\|u\|_{M_1}\|w\|_{\Phi}. \tag{13.14}$$

PROOF. As in the proof of the preceding lemma, we consider the linear operators $A_u w(x) = u(x)w(x)$, $w(x) \in L^*_{\Phi}$, defined by the functions $u(x) \in L^*_{M_1}$, $\|u\|_{M_1} \leqslant 1$. The values of these operators on every fixed element $w(x) \in L^*_{\Phi}$ are bounded since, in virtue of Lemma 13.4, the linear operator $Bu(x) = u(x)w(x)$, acting from $L^*_{M_1}$ into E, is bounded.

According to a known theorem (see, e.g., BANACH [1]), the norms of the operators A_u are uniformly bounded. Let $\|A_u\| \leqslant k$. Then for arbitrary functions $u(x) \in L^*_{M_1}$, $w(x) \in L^*_{\Phi}$, we have that

$$\|uw\|_E = \left\| \frac{u}{\|u\|_{M_1}} w \right\|_E \|u\|_{M_1} = \left\| A \frac{u}{\|u\|_{M_1}} w \right\|_E \|u\|_{M_1} \leqslant$$

$$\leqslant k\|u\|_{M_1}\|w\|_{\Phi}. \text{ ★}$$

This lemma implies, in particular, the following assertion: *If the product* $u(x)v(x)w(x)$ *is summable for any triple of functions* $u(x) \in L^*_{M_1}$, $v(x) \in L^*_{M_2}$, $w(x) \in L^*_{M_3}$, *then the following inequality holds*:

$$\int_G u(x)v(x)w(x)\,dx \leqslant k\|u\|_{M_1}\|v\|_{M_2}\|w\|_{M_3}.$$

This inequality is a generalization of the known Hölder inequality:

$$\int_G u(x)v(x)w(x)\,dx \leqslant$$

$$\leqslant k\left(\int_G |u(x)|^{\alpha_1}dx\right)^{1/\alpha_1}\left(\int_G |v(x)|^{\alpha_2}dx\right)^{1/\alpha_2}\left(\int_G |w(x)|^{\alpha_3}dx\right)^{1/\alpha_3},$$

where $1/\alpha_1 + 1/\alpha_2 + 1/\alpha_3 \leqslant 1$.

This assertion is valid for an arbitrary finite number of factors.

It is easily seen that the product $u(x)w(x)$ will be summable for any functions $u(x) \in L^*_{M_1}$, $w(x) \in L^*_{\Phi}$ only in the case when $\Psi(u) \prec M_1(u)$ or, equivalently, when $N_1(u) \prec \Phi(u)$. Here, as usual, $\Psi(u)$, $N_1(u)$ denote the N-functions which are complementary to $\Phi(u)$, $M_1(u)$, respectively. We note that it is impossible to find an Orlicz space to which all the products $u(x)w(x)$ belong if $\Phi(u) = N_1(u)$. This follows from the following theorem.

THEOREM 13.5. *Suppose $u(x)w(x)$ belongs to some Orlicz space $L^*_{M_2}$ for any pair of functions $u(x) \in L^*_{M_1}$, $w(x) \in L^*_{\Phi}$. Then the N-function $M_1(u)$ increases essentially more rapidly than $\Psi(u)$, or, equivalently, $\Phi(u)$ increases essentially more rapidly than $N_1(u)$.*

PROOF. In virtue of Lemma 13.3, it suffices to show that every bounded set of functions in $L^*_{M_1}$ has equi-absolutely continuous norms in the space L^*_{Ψ}.

Let T be a sphere of radius r in the space $L^*_{M_1}$. In virtue of Lemma 13.5, we have that $\|uw\|_{M_2} \leqslant 2rk$, $u(x) \in T$, $\rho(w; \Phi) \leqslant 1$. It then follows from de la Vallée Poussin's theorem that the functions $u(x)w(x)$ have equi-absolutely continuous integrals, i.e. to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $\left| \int\limits_{G_1} u(x)w(x)\,dx \right| < \varepsilon$ for every $G_1 \subset G$, where mes $G_1 < \delta$. The inequality $\left| \int\limits_{G_1} u(x)w(x)\,dx \right| < \varepsilon$ signifies that mes $G_1 < \delta$ implies that

$$\|u(x)\kappa(x; G_1)\|_{\Psi} = \sup_{\rho(w; \Phi) \leqslant 1} \left| \int\limits_{G_1} u(x)w(x)\,dx \right| < \varepsilon. \ \bigstar$$

THEOREM 13.6. *If for every pair of functions $u(x) \in L^*_{M_1}$, $w(x) \in L^*_{\Phi}$, the product $u(x)w(x)$ belongs to some Orlicz space $L^*_{M_2}$, then the N-functions $M_1(u)$ and $\Phi(u)$ increase essentially more rapidly than the N-function $M_2(u)$.*

PROOF. We shall prove the assertion of the theorem, for example, for the function $M_1(u)$. Let $G_1 \subset G$ and let $\kappa(x; G_1)$ be the characteristic function of the set G_1. Since $\kappa(x; G_1) \in L^*_{\Phi}$, we have that $u(x)\kappa(x; G_1) \in L^*_{M_2}$ and, in virtue of Lemma 13.5, that $\|u(x)\kappa(x; G_1)\|_{M_2} \leqslant k\|u\|_{M_1}\|\kappa\|_{\Phi}$.

Suppose the family of functions $u(x) \in L^*_{M_1}$ has bounded norms, $\|u\|_{M_1} \leqslant a$. Then, in virtue of the preceding inequality and formula

(9.11), we have that

$$\|u(x)\kappa(x; G_1)\|_{M_2} \leqslant ak \operatorname{mes} G_1 \Psi^{-1}(1/\operatorname{mes} G_1),$$

from which it follows that $\lim_{\operatorname{mes} G_1 \to 0} \|u\kappa\|_{M_2} = 0$, i.e. that the functions $u(x)$ have equi-absolutely continuous norms in $L^*_{M_2}$. ★

5. *Sufficient conditions.* We now point out some sufficient conditions for the product of two functions $u(x) \in L^*_{M_1}$, $w(x) \in L^*_{\Phi}$ to belong to an Orlicz space $L^*_{M_2}$.

THEOREM 13.7. *Suppose there exist two mutually complementary N-functions $R(u)$ and $Q(u)$ such that the inequalities*

$$R(u) < M_2^{-1}[M_1(\alpha u)] \tag{13.15}$$

and

$$Q(u) < M_2^{-1}[\Phi(\beta u)] \tag{13.16}$$

are satisfied for $u \geqslant u_0$, where α, β are certain constants. Then, for every pair of functions $u(x) \in L^*_{M_1}$, $w(x) \in L^*_{\Phi}$, the product $u(x)w(x)$ belongs to $L^*_{M_2}$.

PROOF. In virtue of (13.15) and (13.16), the inequalities

$$M_2[R(u)] < M_2[R(u_0)] + M_1(\alpha u) \tag{13.17}$$

and

$$M_2[Q(u)] < M_2[Q(u_0)] + \Phi(\beta u) \tag{13.18}$$

are satisfied for all values of the argument.

Suppose $v(x) \in L_{N_2}$. Then

$$\int_G u(x)w(x)v(x)\,dx = \alpha\beta\|u\|_{M_1}\|w\|_{\Phi} \int_G \frac{u(x)}{\alpha\|u\|_{M_1}} \frac{w(x)}{\beta\|w\|_{\Phi}} v(x)\,dx.$$

Applying Young's inequality twice to the last integral, we obtain that

$$\left|\int_G u(x)w(x)v(x)\,dx\right| \leqslant \alpha\beta\|u\|_{M_1}\|w\|_{\Phi}\left\{\int_G M_2\left[R\left(\frac{u(x)}{\alpha\|u\|_{M_1}}\right)\right]dx + \right.$$

$$\left. + \int_G M_2\left[Q\left(\frac{w(x)}{\beta\|w\|_{\Phi}}\right)\right]dx + 2\int_G N_2[v(x)]\,dx\right\},$$

and, in virtue of (13.17) and (13.18),

$$\left| \int_G u(x)w(x)v(x)\,dx \right| \leqslant \alpha\beta \|u\|_{M_1}\|w\|_{\Phi} \left\{ M_2[R(u_0)] \operatorname{mes} G + \right.$$

$$+ M_2[Q(u_0)] \operatorname{mes} G + \int_G M_1\left[\frac{u(x)}{\|u\|_{M_1}}\right] dx +$$

$$\left. + \int_G \Phi\left[\frac{w(x)}{\|w\|_{\Phi}}\right] dx + 2\int_G N_2[v(x)]\,dx \right\} < \infty.$$

Thus, $u(x)w(x) \in L^*_{M_2}$, with

$$\|uw\|_{M_2} = \sup_{\rho(v;\,N_2)\leqslant 1} \left| \int_G u(x)w(x)v(x)\,dx \right| \leqslant k\|u\|_{M_1}\|w\|_{\Phi}, \qquad (13.19)$$

where $k = \alpha\beta(4 + \{M_2[R(u_0)] + M_2[Q(u_0)]\} \operatorname{mes} G)$. ★

In particular, the assertion introduced above that under the condition $1/\alpha_1 + 1/\alpha_2 \leqslant 1$ the product $u(x)w(x)$ belongs to all L^{γ}, where $1 \leqslant \gamma \leqslant \gamma_0 = \alpha_1\alpha_2/(\alpha_1 + \alpha_2)$, follows from this theorem. In fact, if $M_1(u) = |u|^{\alpha_1}$, $\Phi(u) = |u|^{\alpha_2}$, $M_2(u) = |u|^{\gamma_0}$, then the N-functions $R(u) = M_2^{-1}[M_1(u)] = |u|^{\alpha_1/\gamma_0}$ and $Q(u) = M_2^{-1}[\Phi(u)] = |u|^{\alpha_2/\gamma_0}$ will be mutually complementary since $\gamma_0/\alpha_1 + \gamma_0/\alpha_2 = 1$.

In the sequel, we shall be interested in the case when the N-function $M_1(u)$ satisfies the Δ_3-condition and $M_2(u) = N_1(u)$. As is easily seen, in this case the assertion of Theorem 13.7 is satisfied if the inequality

$$N_1[M_1(u)] < \Phi(\beta u) \qquad (13.20)$$

is valid for large values of u.

In fact, in virtue of Theorem 6.3, we have that $N_1[M_1(u)] < M_1(ku)$ for large values of u, i.e. $M_1(u) < N_1^{-1}[M_1(ku)]$. Setting $M_1(u) = R(u)$, $N_1(u) = Q(u)$, we convince ourselves of the fact that the conditions of Theorem 13.7 are satisfied.

It is also easily seen that in the case when the N-function $M_1(u)$ satisfies the Δ^2-condition and the N-function $M_2(u)$ the Δ_2-condition, then the assertion of Theorem 13.7 is valid if the inequality

$$M_2[N_1(u)] < \Phi(\beta u) \qquad (13.21)$$

is satisfied for large values of the argument. To prove this, one must

set $R(u) = M_1(u)$, $Q(u) = N_1(u)$ and note that an N-function which satisfies the Δ_2-condition increases more slowly than some exponential function.

If the N-function $M_2(u)$ satisfies the Δ'-condition, then another sufficient condition can be given, as follows.

THEOREM 13.8. *Suppose the N-function $M_2(u)$ satisfies the Δ'-condition. Let the inequalities*

$$R(\alpha u) < M_1[M_2^{-1}(u)] \tag{13.22}$$

and

$$Q(\beta u) < \Phi[M_2^{-1}(u)] \tag{13.23}$$

be satisfied for $u \geqslant u_0$, where $R(u)$ and $Q(u)$ are mutually complementary N-functions and α, β are constants. Then, for every pair of functions $u(x) \in L^*_{M_1}$, $w(x) \in L^*_{\Phi}$, the product $u(x)w(x)$ belongs to $L^*_{M_2}$.

PROOF. In virtue of (13.22) and (13.23), the inequalities

$$R[\alpha M_2(u)] < R(\alpha u_0) + M_1(u) \tag{13.24}$$

and

$$Q[\beta M_2(u)] < Q(\beta u_0) + \Phi(u) \tag{13.25}$$

are satisfied for all values of u.

Since the N-function $M_2(u)$ satisfies the Δ'-condition, there exist constants a, a_1, b_1 and C such that

$$M_2(uw) \leqslant a + a_1 M_2(u) + b_1 M_2(w) + C M_2(u) M_2(w) \tag{13.26}$$

for all u, w.

Let $v(x) \in L_{N_2}$. Then

$$\int_G u(x)w(x)v(x)\,dx = \|u\|_{M_1}\|w\|_{\Phi} \int_G \frac{u(x)}{\|u\|_{M_1}} \frac{w(x)}{\|w\|_{\Phi}} v(x)\,dx.$$

Applying the Young inequality to the last integral, we obtain that

$$\left| \int_G u(x)w(x)v(x)\,dx \right| \leqslant$$

$$\leqslant \|u\|_{M_1}\|w\|_{\Phi} \left\{ \int_G M_2\left[\frac{u(x)}{\|u\|_{M_1}} \frac{w(x)}{\|w\|_{\Phi}} \right] dx + \int_G N_2[v(x)]\,dx \right\}.$$

In virtue of (13.26), we have that

$$\int_G M_2\left[\frac{u(x)}{\|u\|_{M_1}}\ \frac{w(x)}{\|w\|_\Phi}\right]dx \leqslant a \operatorname{mes} G + a_1 \int_G M_2\left[\frac{u(x)}{\|u\|_{M_1}}\right]dx +$$

$$+ b_1 \int_G M_2\left[\frac{w(x)}{\|w\|_\Phi}\right]dx + C\int_G M_2\left[\frac{u(x)}{\|u\|_{M_1}}\right] M_2\left[\frac{w(x)}{\|w\|_\Phi}\right]dx.$$

Inequalities (13.22) and (13.23) imply that

$$M_2(u) < \alpha_1 + \frac{1}{\alpha} M_1(u)$$

and

$$M_2(u) < \beta_1 + \frac{1}{\beta}\Phi(u)$$

for all u, where α_1, β_1 are certain constants. Therefore,

$$\int_G M_2\left[\frac{u(x)}{\|u\|_{M_1}}\ \frac{w(x)}{\|w\|_\Phi}\right]dx \leqslant a \operatorname{mes} G + \alpha_1 a_1 \operatorname{mes} G +$$

$$+ \frac{a_1}{\alpha}\int_G M_1\left[\frac{u(x)}{\|u\|_{M_1}}\right]dx + \beta_1 b_1 \operatorname{mes} G + \frac{b_1}{\beta}\int_G \Phi\left[\frac{w(x)}{\|w\|_\Phi}\right]dx +$$

$$+ C\int_G M_2\left[\frac{u(x)}{\|u\|_{M_1}}\right] M_2\left[\frac{w(x)}{\|u\|_\Phi}\right]dx \leqslant (a + \alpha_1 a_1 + \beta_1 b_1) \operatorname{mes} G +$$

$$+ \frac{a_1}{\alpha} + \frac{b_1}{\beta} + C\int_G M_2\left[\frac{u(x)}{\|u\|_{M_1}}\right] M_2\left[\frac{w(x)}{\|w\|_\Phi}\right]dx.$$

In virtue of Young's inequality and inequalities (13.24) and (13.25),

$$\int_G M_2\left[\frac{u(x)}{\|u\|_{M_1}}\right] M_2\left[\frac{w(x)}{\|w\|_\Phi}\right]dx \leqslant \frac{1}{\alpha\beta}\left\{\int_G R\left(\alpha M_2\left[\frac{u(x)}{\|u\|_{M_1}}\right]\right)dx +\right.$$

$$\left. + \int_G Q\left(\beta M_2\left[\frac{w(x)}{\|w\|_\Phi}\right]\right)dx\right\} \leqslant \frac{1}{\alpha\beta}\Big\{[R(\alpha u_0) + Q(\beta u_0)] \operatorname{mes} G +$$

$$+\int_G M_1\left[\frac{u(x)}{\|u\|_{M_1}}\right]dx+\int_G \Phi\left[\frac{w(x)}{\|w\|_{\Phi}}\right]dx\Big\}\leqslant$$

$$\leqslant\frac{1}{\alpha\beta}\{2+[R(\alpha u_0)+Q(\beta u_0)]\operatorname{mes} G\}.$$

Combining all the estimates obtained, we see that

$$\int_G u(x)w(x)v(x)\,dx<\infty$$

for all $v(x)\in L_{N_2}$ with

$$\|uw\|_{M_2}=\sup_{\rho(v;\,N_2)\leqslant 1}\left|\int_G u(x)w(x)v(x)\,dx\right|\leqslant k\|u\|_{M_1}\|w\|_{\Phi}, \quad (13.27)$$

where

$$k=\Big\{1+\frac{2c}{\alpha\beta}+[a+\alpha_1 a_1+\beta_1 b_1+CR(\alpha u_0)+$$

$$+CQ(\beta u_0)]\operatorname{meas} G+\frac{a_1}{\alpha}+\frac{b_1}{\beta}\Big\}.\ \bigstar$$

§ 14. Linear functionals

1. *Linear functionals in L_M^*.* Suppose $M(u)$ and $N(v)$ are mutually complementary N-functions. Let $v(x)$ be a fixed function in L_M^*. It follows from Hölder's inequality (9.16) that the linear functional

$$l(u)=(u,v)=\int_G u(x)v(x)\,dx,\ u(x)\in L_M^*, \quad (14.1)$$

is defined on the entire space L_M^*.

The inequality

$$\|l\|\leqslant\|v\|_N\leqslant 2\|l\| \quad (14.2)$$

holds, where $\|l\|$ denotes the norm of the functional $l(u)$:

$$\|l\|=\sup_{\|u\|_M\leqslant 1}|l(u)|.$$

The left inequality in (14.2) follows from the Hölder inequality:

$$|l(u)| = |(u, v)| \leqslant \|u\|_M \|v\|_N.$$

The right inequality in (14.2) follows from (9.12):

$$\|v\|_N = \sup_{\rho(u;\, M)\leqslant 1} |(u, v)| \leqslant \sup_{\|u\|_M \leqslant 2} |(u, v)| = 2\|l\|.$$

We introduce the notation: $k(v) = \|v\|_N / \|l\|$. Then inequality (14.2) can be rewritten in the form $1 \leqslant k(v) \leqslant 2$.

The function $k(v)$ was studied in detail by D. V. Salehov. The simplest properties of this function are given below.

As an example, we shall compute the value of the function $k(v)$ for the Orlicz spaces defined by the N-functions $M(u) = |u|^\alpha/\alpha$ $(\alpha > 1)$, i.e. for the case of the spaces L^α.

We recall that for every function $u(x) \in L_M$ we have

$$\|u\|_M = \alpha^{1/\alpha}\beta^{1/\beta}\left\{\int_G M[u(x)]\,dx\right\}^{1/\alpha},$$

where $1/\alpha + 1/\beta = 1$. Therefore,

$$\|l\| = \sup_{\|u\|_M \leqslant 1} \left|\int_G u(x)v(x)\,dx\right| =$$

$$= \sup_{\alpha^{1/\alpha}\beta^{1/\beta}\{\int_G M[u(x)]dx\}^{1/\alpha}\leqslant 1} \left|\int_G u(x)v(x)\,dx\right| =$$

$$= \frac{1}{\alpha^{1/\alpha}\beta^{1/\beta}} \sup_{\int_G M[u(x)]dx \leqslant 1} \left|\int_G u(x)v(x)\,dx\right| = \frac{\|v\|_N}{\alpha^{1/\alpha}\beta^{1/\beta}},$$

from which it follows that $k(v) = \alpha^{1/\alpha}\beta^{1/\beta}$, $v(x) \in L_N^*$.

The constant $\alpha^{1/\alpha}\beta^{1/\beta}$ $(1/\alpha + 1/\beta = 1)$ can assume any value in the interval $(1, 2]$.

Let $M(u)$ and $N(v)$ be arbitrary mutually complementary N-functions. Let G_1 be a subset of G. We set $v(x) = N^{-1}(1/\text{mes } G_1)\kappa(x; G_1)$, where $\kappa(x; G_1)$ is the characteristic function of the set G_1. We shall show that

$$k(v) = \text{mes } G_1 M^{-1}(1/\text{mes } G_1) N^{-1}(1/\text{mes } G_1). \tag{14.3}$$

It follows from the formula for the norm of the characteristic

function that $\|v\|_N$ equals the number appearing in the right member of formula (14.3). It therefore suffices to show that the norm of the linear functional (14.1), defined by the function $v(x)$, equals unity.

Since $\rho(v; N) = 1$, we have that

$$\|l\| = \sup_{\|u\|_M \leqslant 1} |(u, v)| \leqslant \sup_{\|u\|_M \leqslant 1} \left\{ \sup_{\rho(w; N) \leqslant 1} |(u, w)| \right\} = 1.$$

On the other hand,

$$\|l\| = \sup_{\|u\|_M \leqslant 1} |(u, v)| \geqslant \left(\frac{\kappa(x; G_1)}{\|\kappa(x; G_1)\|_M}, v \right) = 1.$$

This means that $\|l\| = 1$, and formula (14.3) is proved.

Thus, the set of values of the function $k(v)$ comprises at least the set of values of the function

$$f(\gamma) = \frac{M^{-1}(\gamma) N^{-1}(\gamma)}{\gamma}, \quad \frac{1}{\text{mes } G} < \gamma < \infty.$$

D. V. Salehov showed that $f(\gamma) = \text{constant}$ only in the case when $M(u) = k|u|^\alpha$, $\alpha > 1$. This result signifies that the function $k(v)$ is constant only in the case of the spaces L^α.

We shall prove this assertion for the simplest case — when $f(\gamma) = 2$, $\gamma \geqslant 1/\text{mes } G$.

We note first of all that the equality $f(\gamma) = c$ implies the continuity of the right derivatives $p(u)$ and $q(u)$ of the N-functions $M(u)$ and $N(u)$. In fact, it follows from the equality

$$M^{-1}(\gamma) N^{-1}(\gamma) = c\gamma, \quad \frac{1}{\text{mes } G} \leqslant \gamma < \infty,$$

that

$$\frac{M^{-1}(\gamma)}{q[N^{-1}(\gamma)]} + \frac{N^{-1}(\gamma)}{p[M^{-1}(\gamma)]} = c.$$

Since the derivatives $q(u)$ and $p(u)$ have positive saltuses at their points of discontinuity, it follows from the last equality that $q(u)$ and $p(u)$ are continuous.

Now let $f(\gamma) = 2$, i.e., $M^{-1}(\gamma) N^{-1}(\gamma) = 2\gamma$. In virtue of Young's inequality, we always have that $M^{-1}(\gamma) N^{-1}(\gamma) \leqslant 2\gamma$. We therefore have the case when the equality sign is attained in Young's inequality. In virtue of the continuity of the derivatives $p(u)$ and $q(u)$,

the equality sign is attained if $M^{-1}(\gamma) = q[N^{-1}(\gamma)]$ or, equivalently, if $N^{-1}(\gamma) = p[M^{-1}(\gamma)]$.

Making use of the last equality and the relation $M^{-1}(\gamma)N^{-1}(\gamma) = 2\gamma$, we obtain that

$$\frac{2\gamma}{M^{-1}(\gamma)p[M^{-1}(\gamma)]} = 1$$

for all $\gamma \geqslant 1/\text{mes } G$. We set $M^{-1}(\gamma) = u$. Then $p(u)/M(u) = 2/u$ for $u \geqslant u_0 = M^{-1}(1/\text{mes } G)$. Integrating the equality $p(u)/M(u) = 2/u$ between the limits from u_0 to $|u|$, we obtain that

$$M(u) = \frac{M(u_0)}{u_0^2}|u|^2, \quad u \geqslant u_0.$$

It follows from (14.2) that the space L_N^* can be considered as a linear subset of the space of functionals conjugate to L_M^*. In this connection, the norm of the space L_N^* is equivalent to the norm induced on L_N^* as a subset of the space of functionals. Since the space L_N^* is dense with respect to the Orlicz norm, it forms a closed subspace, L_N^*, in the space of functionals, which, generally speaking, does not coincide with the space of functionals on L_M^*, as is indicated by the following theorem.

THEOREM 14.1. *Suppose the N-function $M(u)$ does not satisfy the Δ_2-condition. Then* (14.1) *is not the general form of a linear functional on L_M^*.*

PROOF. Let E_M be a linear subspace of L_M^* which is the closure in L_M^* of a set of bounded functions. In virtue of Theorem 9.4 and (10.1), E_M is a proper subset of L_M^*. Let $u_0(x) \in L_M^* \setminus E_M$. We define a linear functional $l(u)$ on L_M^* by setting $l(u_0) = 1$ and $l(u) = 0$ for $u(x) \in E_M$, and then extending it by the Hahn-Banach theorem, with preservation of norm, to all of L_M^* (see BANACH [1]).

We assume that this functional $l(u)$ is representable in the form

$$l(u) = \int_G u(x)v(x)\,dx, \quad u(x) \in L_M^*,$$

where $v(x)$ is some function. We form the following sequence of bounded functions:

$$v_n(x) = \begin{cases} v(x) & \text{for } |v(x)| \leqslant n \\ 0 & \text{for } |v(x)| > n. \end{cases} \quad (n = 1, 2, \cdots),$$

By the construction of the functional $l(u)$, we have that

$$\int_G v_n(x)v(x)\,dx = 0 \quad (n = 1, 2, \cdots),$$

from which it follows that the function $v(x)$ equals zero almost everywhere. But then we also have that $l(u_0) = 0$ which contradicts the fact that $l(u_0) = 1$. ★

2. *General form of a linear functional on* E_M.

THEOREM 14.2. *Formula* (14.1), *where* $v(x) \in L_M^*$, *yields the general form of a linear functional on* E_M.

PROOF. We shall carry out the proof of this theorem by following the line of reasoning customary for such theorems.

Let $l(u)$ be a linear functional defined on E_M. We define, on the totality of all measurable subsets $\mathscr{E}$ of the set G, a set function $F(\mathscr{E})$ by means of the equality $F(\mathscr{E}) = l[\kappa(x; \mathscr{E})]$, where $\kappa(x; \mathscr{E})$ is the characteristic function of the set $\mathscr{E}$.

The additive function $F(\mathscr{E})$ is absolutely continuous inasmuch as, in virtue of (9.11), we have that

$$|F(\mathscr{E})| = |l[\kappa(x; \mathscr{E})]| \leqslant \|l\| \operatorname{mes} \mathscr{E} N^{-1}\left(\frac{1}{\operatorname{mes} \mathscr{E}}\right),$$

from which it follows that

$$\lim_{\operatorname{mes} \mathscr{E} \to 0} |F(\mathscr{E})| = 0.$$

In virtue of the Radon-Nikodym theorem (cf. HILLE [1]), the function $F(\mathscr{E})$ is representable in the form

$$F(\mathscr{E}) = \int_{\mathscr{E}} v(x)\,dx, \tag{14.4}$$

where $v(x)$ is a summable function on G.

It follows from (14.4) that the equality

$$l(u) = \int_G u(x)v(x)\,dx \tag{14.5}$$

is also valid for every measurable function $u(x)$ which assumes a finite number of values.

Let $u(x)$ be an arbitrary function in L_M. A sequence of bounded functions $u_n(x)$ $(n = 1, 2, \cdots)$ which converges almost everywhere to $u(x)$ can be found such that $|u_n(x)| \leqslant |u(x)|$ almost everywhere so that $\|u_n\|_M \leqslant \|u\|_M$. The sequence of positive functions $|u_n(x)v(x)|$ also converges almost everywhere to the function $|u(x)v(x)|$. In virtue of Fatou's theorem and (16.5), we have that

$$\left| \int_G u(x)v(x)\,dx \right| \leqslant \sup_n \left\{ \int_G |u_n(x)v(x)|\,dx \right\} =$$

$$= \sup_n |\boldsymbol{l}(|u_n(x)| \operatorname{sgn} v(x))| \leqslant \|\boldsymbol{l}\| \sup_n \|u_n\|_M \leqslant \|\boldsymbol{l}\| \|u\|_M < \infty.$$

This means that $v(x) \in L_N^*$.

We denote the functional $\int_G u(x)v(x)\,dx$, defined on L_M^*, by $\boldsymbol{l}_1(u)$:

$$\boldsymbol{l}_1(u) = \int_G u(x)v(x)\,dx.$$

In virtue of (14.5), the continuous linear functionals $\boldsymbol{l}(u)$ and $\boldsymbol{l}_1(u)$ assume the same values on a set of bounded functions which is everywhere dense in E_M. This signifies that they take on the same values on all of E_M, i.e. formula (14.5) is valid for all functions $u(x) \in E_M$.

Distinct functions $v(x) \in L_N^*$, obviously, generate distinct functionals on E_M. ★

We shall momentarily denote the norm of the functional (14.1), considered only on E_M, by $\|\boldsymbol{l}\|_1$. For every $\varepsilon > 0$, a function $u(x) \in L_M^*$, $\|u\|_M = 1$, can be found such that

$$\int_G u(x)v(x)\,dx \geqslant \|\boldsymbol{l}\| - \varepsilon.$$

We set

$$u_n(x) = \begin{cases} u(x) & \text{for } |u(x)| \leqslant n, \\ 0 & \text{for } |u(x)| > n \end{cases} \qquad (n = 1, 2, \cdots).$$

Clearly, $\|u_n\|_M \leqslant \|u\|_M$ and $u_n(x) \in E_M$. It follows from the

absolute continuity of the integral that

$$\int_G u_n(x)v(x)\,dx \geqslant \int_G u(x)v(x)\,dx - \varepsilon$$

for sufficiently large n, from which it follows that

$$\|l\|_1 \geqslant \|u_n\|_M \|l\|_1 \geqslant \int_G u_n(x)v(x)\,dx \geqslant \|l\| - 2\varepsilon.$$

It follows from the inequality just obtained and the obvious relation $\|l\|_1 \leqslant \|l\|$ that $\|l\|_1 = \|l\|$, i.e. that

$$\|l\| = \sup_{\substack{\|u\|_M \leqslant 1, \\ u \in E_M}} |l(u)|. \tag{14.6}$$

Equality (14.6) allows us to also use the same notation $\|l\|$ when the functional (14.1) is considered on all of L_M^* as when it is considered only on E_M.

If the N-function $M(u)$ satisfies the Δ_2-condition, then E_M coincides with $L_M^* = L_M$. In this case, (14.1) yields the general form of a linear functional on L_M^*.

3. *E_N-weak convergence.* We shall say that the sequence $u_n(x) \in L_M^*$ $(n = 1, 2, \cdots)$ *is E_N-weakly convergent* if the sequence of numbers

$$l(u_n) = \int_G u_n(x)v(x)\,dx \quad (n = 1, 2, \cdots)$$

converges for every function $v(x) \in E_N$.

The definition just introduced differs from the usual definition inasmuch as the functions $v(x) \in E_N$ do not, generally speaking, define all linear functionals on L_M^*. This definition coincides with the usual definition if both N-functions $M(u)$ and $N(v)$ satisfy the Δ_2-condition.

If we consider weak convergence in the space E_N, then the definition introduced above coincides with the usual definition if the N-function $N(v)$ satisfies the Δ_2-condition.

In the general case, E_N-weak convergence is weak convergence in the space L_M^* considered as the space of functionals on E_N. In fact, as was already shown, there exists a linear one-to-one

correspondence between the elements of this space of functionals and the elements of L_M^* in which connection the norms of the corresponding elements are equivalent. Thus, to every E_N-weakly convergent sequence of elements in L_M^* there corresponds a weakly convergent sequence of linear functionals on E_N.

The following assertions follow from general theorems (see, e.g., LYUSTERNIK and SOBOLEV [1]).

THEOREM 14.3. *If the sequence of functions $u_n(x) \in L_M^*$ $(n = 1, 2, \cdots)$ is E_N-weakly convergent, then the norms $\|u_n\|_M$ $(n = 1, 2, \cdots)$ are uniformly bounded.*

THEOREM 14.4. *Every space L_M^* is E_N-weakly complete in the sense that for every E_N-weakly convergent sequence of functions $u_n(x) \in L_M^*$ $(n = 1, 2, \cdots)$ a unique function $u(x) \in L_M^*$ can be found such that*

$$\lim_{n\to\infty} \int_G u_n(x)v(x)\,dx = \int_G u(x)v(x)\,dx, \quad v(x) \in E_N.$$

Every space L_M^* is E_N-weakly compact, i.e. every bounded sequence contains an E_N-weakly convergent subsequence.

We note that the E_N-weak closure of the space E_M in the space L_M^* is the entire space L_M^*. This follows from the fact that for every function $u(x) \in L_M^*$ the sequence of bounded functons

$$u_n(x) = \begin{cases} u(x) & \text{for } |u(x)| \leqslant n, \\ 0 & \text{for } |u(x)| > n \end{cases} \qquad (n = 1, 2, \ldots)$$

converges E_N-weakly to $u(x)$ inasmuch as

$$\lim_{n\to\infty} \int_G [u(x) - u_n(x)]v(x)\,dx = 0$$

for all $v(x) \in E_N$ (and also for all $v(x) \in L_N^*$).

As was already shown (see p. 91), the norms of the functions $u_n(x)$ also converge to $\|u\|_M$. Thus, the sequence $u_n(x)$ converges almost everywhere to $u(x)$, is E_N-weakly convergent to $u(x)$, and $\|u_n\|_M \to \|u\|_M$. However, generally speaking, $u_n(x)$ does not converge in norm to $u(x)$.

Every sequence of functions which converges with respect to the norm in L_M^*, obviously, is E_N-weakly convergent. The converse,

of course, does not hold. We shall use the following obvious assertion in the sequel.

THEOREM 14.5. *If the sequence of functions* $u_n(x) \in L_M^*$ $(n = 1, 2, \cdots)$ *is* E_N*-weakly convergent and compact in the sense of convergence in norm in* L_M^*, *then it also converges in norm.*

The following criterion for E_N-weak convergence is sometimes convenient to use.

THEOREM 14.6. *Suppose the sequence* $u_n(x) \in L_M^*$ $(n = 1, 2, \cdots)$ *converges in measure to the function* $u(x)$ *with the norms* $\|u_n\|_M$ $(n = 1, 2, \cdots)$ *uniformly bounded. Then* $u(x) \in L_M^*$ *and the sequence* $u_n(x)$ $(n = 1, 2, \cdots)$ *is* E_N*-weakly convergent to* $u(x)$.

PROOF. In virtue of the fact that a sphere in L_M^* is E_N-weakly compact, every sequence of functions in L_M^* which are bounded in norm contains an E_N-weakly convergent subsequence. It is therefore sufficient in our case to show that for any subsequence $u_{n_k}(x)$ which is E_N-weakly convergent in L_M^* to $u_0(x) \in L_M^*$, we have $u_0(x) = u(x)$.

We denote by $\kappa_m(x)$ the characteristic function of some fixed set of points on which $|u(x) - u_0(x)| \leqslant m$, and the function $\operatorname{sgn}[u(x) - u_0(x)]$ by $v_0(x)$.

Suppose $\varepsilon > 0$ is prescribed. Since the functions $u_0(x)$, $u_{n_k}(x)$ $(k = 1, 2, \cdots)$ have, in virtue of the de la Vallée Poussin theorem (see § 11, subsection 1), equi-absolutely continuous integrals, a $\delta > 0$ can be found such that

$$\int_{\mathscr{E}} |u_0(x)|\,dx < \frac{\varepsilon}{5}, \quad \int_{\mathscr{E}} |u_{n_k}(x)|\,dx < \frac{\varepsilon}{5} \quad (k = 1, 2, \cdots)$$

provided $\operatorname{mes} \mathscr{E} < \delta$ $(\mathscr{E} \subset G)$. We shall assume that $\delta < \varepsilon/(5m)$. It follows from the convergence in measure of the subsequence $u_{n_k}(x)$ to the function $u(x)$ and the E_N-weak convergence of this sequence to the function $u_0(x)$ that there exists a k_0 such that, for $k > k_0$,

$$\int_G [u_{n_k}(x) - u_0(x)]v_0(x)\kappa_m(x)\,dx < \frac{\varepsilon}{5}$$

and $\operatorname{mes} G_k < \delta$, where

$$G_k = G\{|u_{n_k}(x) - u(x)| \geqslant \varepsilon/(5 \operatorname{mes} G)\}.$$

Then, for $k > k_0$, we have that

$$\int_G |u(x) - u_0(x)|\kappa_m(x)\,dx \leqslant \left| \int_G [u_{n_k}(x) - u_0(x)]v_0(x)\kappa_m(x)\,dx \right| +$$

$$+ \int_{G \setminus G_k} |u(x) - u_{n_k}(x)|\,dx + \int_{G_k} |u_{n_k}(x)|\,dx +$$

$$+ \int_{G_k} |u_0(x)|\,dx + \int_{G_k} |u(x) - u_0(x)|\kappa_m(x)\,dx <$$

$$< \frac{\varepsilon}{5} + \frac{\varepsilon}{5\ \text{mes}\ G}\ \text{mes}\ (G \setminus G_k) + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + m\ \text{mes}\ G_k < \varepsilon.$$

Since ε is arbitrary, we have that

$$\int_G |u(x) - u_0(x)|\kappa_m(x)\,dx = 0,$$

i.e. $u_0(x) = u(x)$ almost everywhere. ★

4. *E_N-weakly continuous linear functionals.* We shall say that the linear functional $\boldsymbol{l}(u)$ is *E_N-weakly continuous* on L_M^* if for every sequence $u_n(x) \in L_M^*$ $(n = 1, 2, \cdots)$ which is E_N-weakly convergent to the element $u_0(x) \in L_M^*$ the equality $\lim_{n\to\infty} \boldsymbol{l}(u_n) = \boldsymbol{l}(u_0)$ holds.

It follows from this definition that functionals of the form (14.1), where $v(x) \in E_N$, are E_N-weakly continuous. The converse assertion also holds.

THEOREM 14.7. *Every E_N-weakly continuous linear functional on L_M^* is representable in the form* (14.1), *where* $v(x) \in E_N$.

PROOF. We consider the space $\bar{E}_N$ of functionals defined on the space E_N. In virtue of Theorem 14.2 on the general form of a linear functional in E_N, there exists a one-to-one correspondence, defined by the formula

$$\boldsymbol{f}(v) = \int_G v(x)u(x)\,dx,\ u(x) \in L_M^*,\ v(x) \in E_N, \tag{14.7}$$

between the functionals $\boldsymbol{f} \in \bar{E}_N$ and the functions $u(x) \in L_M^*$. In this connection, in virtue of the definition of E_N-weak convergence, to the sequence of functions $u_n(x) \in L_M^*$ which is E_N-

weakly convergent to the function $u(x) \in L_M^*$ there corresponds in one-to-one fashion a sequence of functionals

$$f_n(v) = \int_G v(x)u_n(x)\,dx$$

which converges weakly to the functional

$$f(v) = \int_G v(x)u(x)\,dx.$$

Let $l_0(u)$ be an E_N-weakly continuous functional defined on the space L_M^*. We define a functional $\Phi(f)$, $f \in \bar{E}_N$, on the space $\bar{E}_N$ by means of the equality

$$\Phi(f) = l_0(u), \tag{14.8}$$

where $u(x) \in L_M^*$ is the function corresponding to the functional f in virtue of the one-to-one correspondence established between the functionals on E_N and the functions in the space L_M^*.

It is obvious that the E_N-weak continuity of the functional $l_0(u)$ on L_M^* implies the weak continuity of the functional $\Phi(f)$ on $\bar{E}_N$. Since the space E_N is separable, we have, in virtue of Banach's theorem on the general form of a weakly continuous functional in the conjugate space, that there exists a function $v_0(x) \in E_N$ such that $\Phi(f) = f(v_0)$ for all $f \in E_N$, i.e.

$$l_0(u) = \int_G v_0(x)u(x)\,dx$$

for all $u(x) \in L_M^*$. [Banach's theorem (see BANACH [1]) asserts that *if E is a separable space and $\Phi(f)$ is a weakly continuous functional on the space $\bar{E}$ conjugate to E, then there exists an element $v_0 \in E$ such that $\Phi(f) = f(v_0)$ for all $f \in \bar{E}$.*] ★

5. *Norm of a functional and* $\|v\|_{(N)}$. If we consider the Luxemburg norm $\|u\|_{(M)}$ (see § 9, subsection 7) in the space L_M^*, then the norm of every linear functional $l(u)$, which admits of an integral representation (14.1), is defined by the equality

$$\|l\|_{(M)} = \sup_{\|u\|_{(M)} \leqslant 1} \left| \int_G u(x)v(x)\,dx \right|,$$

and, in virtue of (9.25), $\|l\|_{(M)} = \|v\|_{(N)}$.

The equality

$$\|\boldsymbol{l}\| = \|v\|_{(N)} \tag{14.9}$$

also holds, where $\|\boldsymbol{l}\|$ denotes the usual norm of the functional (14.1). This equality coincides with the equality

$$\|v\|_{(N)} = \sup_{\|u\|_M \leqslant 1} \left| \int_G u(x)v(x)\,dx \right|. \tag{14.10}$$

To prove (14.10), we note that (9.26) implies the inequality

$$\sup_{\|u\|_M \leqslant 1} \left| \int_G u(x)v(x)\,dx \right| \leqslant \|v\|_{(N)}.$$

Thus, it will suffice to show that

$$\sup_{\|u\|_M \leqslant 1} \left| \int_G u(x)v(x)\,dx \right| \geqslant \|v\|_{(N)}. \tag{14.11}$$

We denote by T the unit sphere $\|v\|_{(N)} \leqslant 1$ considered as a subset of the space L_1 of summable functions on G.

The convex set T is closed with respect to the norm in L_1 inasmuch as

$$\lim_{n\to\infty} \int_G |v_n(x) - w(x)|\,dx = 0, \quad v_n(x) \in T$$

implies that $v_n(x)$ converges in measure to $w(x)$, in which connection, in virtue of Fatou's theorem and (9.21), we have that

$$\int_G N[w(x)]\,dx \leqslant \sup_n \int_G N[v_n(x)]\,dx \leqslant 1.$$

Let $v_0(x)$ be a fixed non-zero function in the space L_N^*. Clearly, the function $(1 + \varepsilon)v_0(x)/\|v_0\|_{(N)}$, where $\varepsilon > 0$, is not in T. As we know (see BANACH [1]), a linear functional $\boldsymbol{f}(v)$, defined on L_1, can be constructed such that

$$\boldsymbol{f}\left[(1 + \varepsilon)\frac{v_0(x)}{\|v_0\|_{(N)}}\right] > \boldsymbol{f}(v), \quad v(x) \in T. \tag{14.12}$$

The functional $f(v)$ admits of the integral representation

$$f(v) = \int_G v(x)h(x)\,dx, \quad v(x) \in L_1,$$

where $h(x)$ is an essentially bounded function (in this connection, see AHIEZER and GLAZMAN [1]). It therefore follows from (14.12) that

$$\frac{1+\varepsilon}{\|v_0\|_{(N)}} \int_G v_0(x)h(x)\,dx \geqslant \sup_{v \in T} \int_G v(x)h(x)\,dx =$$

$$= \sup_{\|v\|_{(N)} \leqslant 1} \int_G |v(x)|\,|h(x)|\,dx$$

and, in virtue of (9.25),

$$\frac{1+\varepsilon}{\|v_0\|_{(N)}} \int_G v_0(x)h(x)\,dx \geqslant \|h\|_M.$$

It follows that

$$(1+\varepsilon) \int_G v_0(x) \frac{h(x)}{\|h\|_M}\,dx \geqslant \|v_0\|_{(N)},$$

and, since ε is arbitrary, we have that

$$\int_G v_0(x) \frac{h(x)}{\|h\|_M}\,dx \geqslant \|v_0\|_{(N)}.$$

Now, this inequality implies (14.11).

Chapter III

OPERATORS IN ORLICZ SPACES

§ 15. Conditions for the continuity of linear integral operators

1. *Formulation of the problem.* This entire chapter will be devoted to the study of linear operators $\boldsymbol{A}$ operating from one Orlicz space $L^*_{M_1}$ into another Orlicz space $L^*_{M_2}$. We shall denote the class of linear operators operating from the space B_1 into the space B_2 by $\{B_1 \to B_2\}$. The class of continuous operators will be denoted by $\{B_1 \to B_2;\ \text{c.}\}$ and the class of completely continuous operators will be denoted by $\{B_1 \to B_2;\ \text{comp. c.}\}$.

Basically, we shall be interested in integral operators of the form

$$\boldsymbol{A}u(x) = \int_G k(x, y)u(y)\,dy. \tag{15.1}$$

The fundamental problem of the present section consists in elucidating the conditions under which the operator (15.1) is continuous considered as an operator operating from $L^*_{M_1}$ into $L^*_{M_2}$, i.e. that it satisfies the condition

$$\|\boldsymbol{A}u\|_{M_2} \leqslant \|\boldsymbol{A}\|\,\|u\|_{M_1},$$

where $\|\boldsymbol{A}\|$ is some number.

We shall naturally search for conditions for the continuity of $\boldsymbol{A}$ in the various characteristics of the kernel $k(x, y)$. The most suitable such characteristic is that the kernel belongs to some Orlicz space, i.e. the finiteness of the integral

$$\int_G \int_G \Psi[\alpha k(x, y)]\,dx\,dy$$

for some α.

Below, we denote by $\hat{G}$ the topological product $G \times G$ equipped with the natural measure. By $\hat{L}_M$, $\hat{L}^*_M$, $\hat{E}_M$ we will denote the corresponding class and spaces $L_M(\hat{G})$, $L^*_M(\hat{G})$ and $E_M(\hat{G})$.

2. *General theorem.* As usual, we shall denote by $N_1(v)$ and $N_2(v)$ the complementary functions to the given N-functions $M_1(u)$ and $M_2(u)$.

THEOREM 15.1. *Let $\Phi(u)$ be an N-function such that for $u(x) \in L^*_{M_1}$, $v(x) \in L^*_{N_2}$ we have*

$$w(x, y) = u(y)v(x) \in \hat{L}^*_{\hat{\Phi}}, \tag{15.2}$$

with

$$\|w(x, y)\|_{\hat{\Phi}} \leqslant l\|u\|_{M_1}\|v\|_{N_2}, \tag{15.3}$$

where l is a constant. Suppose the kernel $k(x, y)$ of the linear integral operator (15.1) *belongs to the space $\hat{L}^*_{\hat{\Psi}}$, where $\Psi(v)$ is the complementary N-function to the N-function $\Phi(u)$. Then the operator* (15.1) *belongs to $\{L^*_{M_1} \to L^*_{M_2}; \text{c.}\}$.*

PROOF. In virtue of Hölder's inequality and (15.3) we have, for $u(x) \in L^*_{M_1}$, $v(x) \in L_{N_2}$, that

$$\int_G \boldsymbol{A}u(x)v(x)\,dx = \int_G \int_G k(x, y)u(y)v(x)\,dx\,dy \leqslant$$

$$\leqslant \|k(x, y)\|_{\hat{\Psi}}\|w(x, y)\|_{\hat{\Phi}} \leqslant l\|k(x, y)\|_{\hat{\Psi}}\|v\|_{N_2}\|u\|_{M_1}, \tag{15.4}$$

from which it follows that the operator $\boldsymbol{A}$ acts from $L^*_{M_1}$ into $L^*_{M_2}$.

Since $\|v\|_{N_2} \leqslant 2$ for $\rho(v; N_2) \leqslant 1$, it follows from (15.4) that

$$\|\boldsymbol{A}u\|_{M_2} = \sup_{\rho(v;\, N_2)\leqslant 1} \left| \int_G \boldsymbol{A}u(x)v(x)\,dx \right| \leqslant 2l\|k(x, y)\|_{\hat{\Psi}}\|u\|_{M_1}. \tag{15.5}$$

Thus, the operator $\boldsymbol{A}$ is bounded and, consequently, it is continuous. ★

From inequality (15.5), we obtain an estimate for the norm of the operator $\boldsymbol{A}$:

$$\|\boldsymbol{A}\| = \sup_{\|u\|_{M_1}\leqslant 1} \|\boldsymbol{A}u\|_{M_2} \leqslant 2l\|k(x, y)\|_{\hat{\Psi}}. \tag{15.6}$$

This estimate is, of course, too high. It can be sharpened in many cases. One way of sharpening the estimate of the norm of the operator $\boldsymbol{A}$ can be based on the application of the strengthened Hölder inequality (9.26).

3. *Existence of the function $\Phi(u)$.* The application of Theorem 15.1 requires knowledge of a function $\Phi(u)$ for which conditions (15.2) and (15.3) are satisfied.

LEMMA 15.1. *Let $\Phi(u)$ be defined as the complementary N-function to the N-function*

$$\Psi(v) = M_2[N_1(v)]. \tag{15.7}$$

Then conditions (15.2) *and* (15.3) *are satisfied.*

PROOF. Let $u(x) \in L^*_{M_1}$, $v(x) \in L^*_{N_2}$. We shall first show that condition (15.2) is satisfied, i.e. that the function $w(x, y) = u(y)v(x)$ belongs to the space $\hat{L}^*_{\Phi}$.

Let $g(x, y) \in \hat{L}_{\Psi}$. Since

$$\left| \int\int_G w(x, y)g(x, y)\,dx\,dy \right| \leqslant$$

$$\leqslant \|u\|_{M_1}\|v\|_{N_2} \int\int_G |g(x, y)| \frac{|u(y)|\,|v(x)|}{\|u\|_{M_1}\|v\|_{N_2}}\,dx\,dy,$$

we have, in virtue of Young's inequality (2.6), applied to the first of the two factors appearing under the integral sign in the right member, that

$$\left| \int\int_G w(x, y)g(x, y)\,dx\,dy \right| \leqslant$$

$$\leqslant \|u\|_{M_1}\|v\|_{N_2}\left\{ \int\int_G N_1[g(x, y)] \frac{|v(x)|}{\|v\|_{N_2}}\,dx\,dy + \right.$$

$$\left. + \int\int_G M_1\left[\frac{u(y)}{\|u\|_{M_1}}\right] \frac{|v(x)|}{\|v\|_{N_2}}\,dx\,dy \right\}.$$

Applying, once more, Young's inequality to the first term in the curly brackets, we obtain that

$$\left| \int\int_G w(x, y)g(x, y)\,dx\,dy \right| \leqslant$$

$$\leqslant \|u\|_{M_1}\|v\|_{N_2}\left\{ \int\int_G M_2[N_1[g(x, y)]]\,dx\,dy + \right.$$

$$\left. + \int\int_G N_2\left[\frac{v(x)}{\|v\|_{N_2}}\right] dx\,dy + \int_G M_1\left[\frac{u(y)}{\|u\|_{M_1}}\right] dy \int_G \frac{|v(x)|}{\|v\|_{N_2}}\,dx \right\}. \tag{15.8}$$

Since

$$\int_G N_2\left[\frac{v(x)}{\|v\|_{N_2}}\right]dx \leqslant 1, \quad \int_G M_1\left[\frac{u(y)}{\|u\|_{M_1}}\right]dy \leqslant 1$$

and the function $v(x)$ is summable, it follows from (13.8) that

$$\left|\int_{\hat{G}}\int w(x, y)g(x, y)\,dx\,dy\right| \leqslant \|u\|_{M_1}\|v\|_{N_2}\left\{\int_{\hat{G}}\int \Psi[g(x, y)]\,dx\,dy + \right.$$

$$\left. + \operatorname{mes} G + \int_G \frac{|v(x)|}{\|v\|_{N_2}}\,dx\right\}, \qquad (15.9)$$

which implies (15.2). In virtue of Young's inequality, we have that

$$\int_G \frac{|v(x)|}{\|v\|_{N_2}}\,dx \leqslant \int_G N_2\left[\frac{v(x)}{\|v\|_{N_2}}\right]dx +$$

$$+ M_2(1) \operatorname{mes} G \leqslant 1 + M_2(1) \operatorname{mes} G.$$

Therefore, if $\rho(g(x, y); \Psi) \leqslant 1$, then it follows from (15.9) that

$$\|w(x, y)\|_{\hat{\Phi}} = \sup_{\rho(g;\,\Psi)\leqslant 1}\left|\int_{\hat{G}}\int w(x, y)g(x, y)\,dx\,dy\right| \leqslant l\|u\|_{M_1}\|v\|_{N_2},$$

where

$$l = 2 + \operatorname{mes} G + M_2(1) \operatorname{mes} G. \bigstar \qquad (15.10)$$

The next lemma is proved in an analogous manner.

LEMMA 15.2. *Let $\Phi(u)$ be defined as the complementary N-function to the N-function*

$$\Psi(v) = N_1[M_2(v)]. \qquad (15.11)$$

Then conditions (15.2) *and* (15.3) *are satisfied.*

Under the conditions of Lemma 15.2, the constant l is defined by means of the equality

$$l = 2 + \operatorname{mes} G + N_1(1) \operatorname{mes} G. \qquad (15.12)$$

4. *Concerning a property of N-functions which satisfy the Δ'-condition.* In the investigation of linear integral operators, an important role is played by N-functions $M(u)$ which are such that $u(x)$, $v(x) \in L^*_M$ implies that $u(y)v(x) \in \hat{L}^*_M$. In this subsection, we elucidate what N-functions possess this property. Clearly, the N-functions $M(u) = k|u|^\alpha$ ($\alpha > 1$) possess the indicated property.

LEMMA 15.3. *Suppose* $u(x)$, $v(x) \in L_M^*$ *implies that the function* $w(x, y) = u(y)v(x)$ *belongs to* L_M^*. *Then the N-function* $M(u)$ *satisfies the* Δ_2*-condition.*

PROOF. Let us assume that the N-function $M(u)$ does not satisfy the Δ_2-condition. Then a monotonic indefinitely increasing sequence of numbers u_n, $u_n \to \infty$, can be found such that

$$M(2u_n) > 2^{2n} M(2^n) M(u_n) \quad (n = 1, 2, \cdots). \tag{15.13}$$

We construct disjoint sets $G_n \subset G$ such that $\operatorname{mes} G_n = M(2) \cdot \cdot \operatorname{mes} G/\{2^n M(2^n)\}$ $(n = 1, 2, \cdots)$ and disjoint sets $\mathscr{E}_n \subset G$ such that $\operatorname{mes} \mathscr{E}_n = M(u_1) \operatorname{mes} G/\{2^n M(u_n)\}$ $(n = 1, 2, \cdots)$.

We set

$$v(x) = \begin{cases} 2^n & \text{if } x \in G_n \quad (n = 1, 2, \cdots), \\ 0 & \text{if } x \in \overline{\bigcup_{n=1}^{\infty}} G_n \end{cases}$$

and

$$u(x) = \begin{cases} u_n & \text{if } x \in \mathscr{E}_n \quad (n = 1, 2, \cdots), \\ 0 & \text{if } x \in \overline{\bigcup_{n=1}^{\infty}} \mathscr{E}_n. \end{cases}$$

Then

$$\int_G M[v(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M[v(x)]\,dx = \sum_{n=1}^{\infty} M(2^n) \operatorname{mes} G_n =$$

$$= M(2) \operatorname{mes} G < \infty,$$

$$\int_G M[u(x)]\,dx = \sum_{n=1}^{\infty} \int_{\mathscr{E}_n} M[u(x)]\,dx = \sum_{n=1}^{\infty} M(u_n) \operatorname{mes} \mathscr{E}_n =$$

$$= M(u_1) \operatorname{mes} G < \infty.$$

On the other hand, for arbitrarily large k, we have that

$$\int\int_G M\left[\frac{u(y)v(x)}{k}\right] dx\,dy = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int\int_{G_i\ \mathscr{E}_j} M\left[\frac{u(y)v(x)}{k}\right] dx\,dy =$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} M\left(\frac{2^i u_j}{k}\right) \operatorname{mes} G_i \operatorname{mes} \mathscr{E}_j \geqslant$$

$$\geqslant \sum_{n=1}^{\infty} M\left(\frac{2^n u_n}{k}\right) \operatorname{mes} G_n \operatorname{mes} \mathscr{E}_n \geqslant \sum_{n=m}^{\infty} M(2u_n)\, \frac{M(2)M(u_1)(\operatorname{mes} G)^2}{2^{2n} M(2^n) M(u_n)},$$

where $2^{m-1} > k$. It follows from this, in virtue of (15.13), that

$$\int\int_G M\left[\frac{u(y)v(x)}{k}\right] dx\,dy = \infty.$$

Thus, $u(x),\ v(x) \in L_M^*$, whereas $u(y)v(x) \notin \hat{L}_M^*$. ★

THEOREM 15.2. *A necessary and sufficient condition that for an arbitrary pair of functions $u(x),\ v(x) \in L_M^*$ the product $w(x, y) = u(y)v(x)$ belong to the space $\hat{L}_M^*$ is that the N-function $M(u)$ satisfy the Δ'-condition.*

Proof of sufficiency. Suppose the N-function $M(u)$ satisfies the Δ'-condition, i.e. there exist positive constants u_0 and C such that

$$M(uv) \leqslant CM(u)M(v) \qquad (u, v \geqslant u_0). \tag{15.14}$$

In virtue of Lemma 5.1, the function $M(u)$ satisfies the Δ_2-condition. Therefore $L_M^* = L_M$.

Let $u(x),\ v(x) \in L_M$. We denote by G_u (respectively G_v) the set $G\{|u(x)| \geqslant u_0\}$ (respectively $G\{|v(x)| \geqslant u_0\}$), where u_0 is the number appearing in condition (15.14). Then, in virtue of this condition, for $x \in G_v,\ y \in G_u$, we have that $M[u(y)v(x)] \leqslant CM[u(y)]M[v(x)]$. From this inequality and the obvious equality

$$\begin{aligned}\int\int_G M[u(y)v(x)]\,dx\,dy &= \\ &= \int_{G_u}\int_{G_v} M[u(y)v(x)]\,dx\,dy + \int_{G\setminus G_u}\int_{G_v} M[u(y)v(x)]\,dx\,dy + \\ &+ \int_{G_u}\int_{G\setminus G_v} M[u(y)v(x)]\,dx\,dy + \int_{G\setminus G_u}\int_{G\setminus G_v} M[u(y)v(x)]\,dx\,dy\end{aligned}$$

it follows that

$$\begin{aligned}\int\int_G M[u(y)v(x)]\,dx\,dy &\leqslant \\ &\leqslant C\int_G M[u(y)]\,dy \int_G M[v(x)]\,dx + \operatorname{mes} G \int_G M[u_0 v(x)]\,dx + \\ &+ \operatorname{mes} G \int_G M[u_0 u(y)]\,dy + M(u_0^2)(\operatorname{mes} G)^2,\end{aligned} \tag{15.15}$$

and since $u_0 v(x)$, $u_0 u(x) \in L_M$, we have $u(y)v(x) \in \hat{L}_M = \hat{L}^*_M$.

Proof of necessity. Let the function $w(x, y) = u(y)v(x) \in \hat{L}^*_M$ if $u(x)$, $v(x) \in L^*_M$. In virtue of Lemma 15.3, the N-function $M(u)$ satisfies the Δ_2-condition. Therefore $\hat{L}^*_M = \hat{L}_M$.

Let us assume that the N-function $M(u)$ does not satisfy the Δ'-condition. Then monotonically indefinitely increasing sequences of positive numbers u_n, v_n $(n = 1, 2, \cdots)$ can be found such that

$$M(u_n v_n) > 2^{2n} M(u_n) M(v_n) \quad (n = 1, 2, \cdots). \tag{15.16}$$

We construct sets $G_n \subset G$ and $\mathscr{E}_n \subset G$ for which

$$\operatorname{mes} G_n = \frac{M(u_1) \operatorname{mes} G}{2^n M(u_n)}, \quad \operatorname{mes} \mathscr{E}_n = \frac{M(v_1) \operatorname{mes} G}{2^n M(v_n)} \quad (n = 1, 2, \cdots),$$

with $G_i \cap G_j = 0$, $\mathscr{E}_i \cap \mathscr{E}_j = 0$ $(i \neq j)$. We set

$$u(x) = \begin{cases} u_n & \text{if } x \in G_n \quad (n = 1, 2, \cdots), \\ 0 & \text{if } x \bar{\in} \bigcup_{n=1}^{\infty} G_n \end{cases}$$

and

$$v(x) = \begin{cases} v_n & \text{if } x \in \mathscr{E}_n \quad (n = 1, 2, \cdots), \\ 0 & \text{if } x \bar{\in} \bigcup_{n=1}^{\infty} \mathscr{E}_n. \end{cases}$$

Then

$$\int_G M[u(x)]\,dx = \sum_{n=1}^{\infty} \int_{G_n} M[u(x)]\,dx = \sum_{n=1}^{\infty} M(u_n) \operatorname{mes} G_n =$$

$$= M(u_1) \operatorname{mes} G < \infty$$

and

$$\int_G M[v(x)]\,dx = \sum_{n=1}^{\infty} \int_{\mathscr{E}_n} M[v(x)]\,dx = \sum_{n=1}^{\infty} M(v_n) \operatorname{mes} \mathscr{E}_n =$$

$$= M(v_1) \operatorname{mes} G < \infty.$$

On the other hand,

$$\int_G \int M[u(y)v(x)]\,dx\,dy = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_{G_i} \int_{\mathscr{E}_j} M[u(y)v(x)]\,dx\,dy =$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} M(u_i v_j) \operatorname{mes} G_i \operatorname{mes} \mathscr{E}_j \geqslant \sum_{n=1}^{\infty} M(u_n v_n) \frac{M(u_1) M(v_1) (\operatorname{mes} G)^2}{2^{2n} M(u_n) M(v_n)},$$

and, in virtue of (15.16),

$$\int_G \int M[u(y)v(x)]\,dx\,dy = \infty.$$

Thus, $u(x) \in L_M$, $v(x) \in L_M$, and $w(x, y) = u(y)v(x) \overline{\in} \hat{L}_M = \hat{L}^*_M$. We have therefore arrived at a contradiction. ★

It follows from the theorem just proved that "$u(x) \in L^*_M$, $v(x) \in L^*_M$ implies $u(y)v(x) \in \hat{L}^*_M$" is not true for all N-functions (even those satisfying the Δ_2-condition).

LEMMA 15.4. *Suppose the N-function $M(u)$ satisfies the Δ'-condition. Then a constant $a > 0$ can be found such that*

$$\|u(y)v(x)\|_M \leqslant a\|u\|_M\|v\|_M \tag{15.17}$$

for $u(x)$, $v(x) \in L_M$.

PROOF. Suppose $M(uv) \leqslant CM(u)M(v)$ for $u,\ v \geqslant u_0 > 1$. In virtue of (15.15),

$$\int_G \int M\left[\frac{u(y)v(x)}{u_0\|u\|_M \cdot u_0\|v\|_M}\right] dx\,dy \leqslant$$

$$\leqslant C \int_G M\left[\frac{u(y)}{\|u\|_M}\right] dy \int_G M\left[\frac{v(x)}{\|v\|_M}\right] dx +$$

$$+ \operatorname{mes} G\left\{\int_G M\left[\frac{u(y)}{\|u\|_M}\right] dy + \int_G M\left[\frac{v(x)}{\|v\|_M}\right] dx\right\} + M(u_0^2)(\operatorname{mes} G)^2,$$

and, since $\rho(u/\|u\|_M; M) \leqslant 1$, $\rho(v/\|v\|_M; M) \leqslant 1$, we have that

$$\int_G \int M\left[\frac{u(y)v(x)}{u_0^2\|u\|_M \cdot \|v\|_M}\right] dx\,dy \leqslant C + 2\operatorname{mes} G + M(u_0^2)(\operatorname{mes} G)^2.$$

Consequently,

$$\left\| \frac{u(y)v(x)}{u_0^2\|u\|_M\|v\|_M} \right\|_M \leqslant 1 + \int_G \int M\left[\frac{u(y)v(x)}{u_0^2\|u\|_M\|v\|_M}\right] dx\,dy \leqslant$$

$$\leqslant 1 + C + 2\operatorname{mes} G + M(u_0^2)(\operatorname{mes} G)^2.$$

Inequality (15.17), in which

$$a = u_0^2[1 + C + 2 \operatorname{mes} G + M(u_0^2)(\operatorname{mes} G)^2], \qquad (15.18)$$

follows from this. ★

Let $Q(u)$ and $R(u)$ be two N-functions. We recall that the relation $Q(u) \prec R(u)$ signifies the existence of positive constants k and u_0 such that $Q(u) \leqslant R(ku)$ $(u \geqslant u_0)$. In virtue of Theorem 13.1, the relation $Q(u) \prec R(u)$ is equivalent to the inclusion $L_R^* \subset L_Q^*$. We also recall (cf. Theorem 13.3) that the inclusion $L_R^* \subset L_Q^*$ implies the existence of a constant $q > 0$ such that $\|u\|_Q \leqslant q\|u\|_R$ $(u(x) \in L_R^*)$.

THEOREM 15.3. *Suppose the N-function $\Phi(u)$ satisfies the Δ'-condition with*

$$\Phi(u) \prec M_1(u), \ \Phi(u) \prec N_2(u). \qquad (15.19)$$

Then conditions (15.2) *and* (15.3) *are satisfied.*

PROOF. Let $u(x) \in L_{M_1}^*$, $v(x) \in L_{N_2}^*$. In virtue of (15.19), we have that $u(x) \in L_\Phi$, $v(x) \in L_\Phi$. Then, in virtue of Lemma 15.4, $w(x, y) = u(z)v(x) \in \hat{L}_\Phi$ and $\|u(y)v(x)\|_\Phi \leqslant a\|u\|_\Phi\|v\|_\Phi$.

It follows from (15.19) that there exist constants q_1 and q_2 such that $\|u\|_\Phi \leqslant q_1\|u\|_{M_1}$ $(u(x) \in L_{M_1}^*)$, $\|v\|_\Phi \leqslant q_2\|v\|_{N_2}$ $(v(x) \in L_{N_2}^*)$. Therefore, $\|u(y)v(x)\|_\Phi \leqslant l\|u\|_{M_1}\|v\|_{N_2}$, where $l = aq_1q_2$. ★

5. *Sufficient conditions for continuity.*

THEOREM 15.4 (FUNDAMENTAL THEOREM ON CONTINUITY). *Let $\Phi(u)$ and $\Psi(v)$ be mutually complementary N-functions. Suppose the kernel $k(x, y)$ of the linear integral operator* (15.1) *belongs to the space $\hat{L}_\Psi^*$. Then the operator* (15.1) *belongs to $\{L_{M_1}^* \to L_{M_2}^*;$ c.$\}$ if any one of the following conditions is satisfied*:

a) $$M_2[N_1(v)] \prec \Psi(v), \qquad (15.20)$$

b) $$N_1[M_2(v)] \prec \Psi(v), \qquad (15.21)$$

c) *the function $\Phi(u)$ satisfies the Δ'-condition and*

$$N_1(v) \prec \Psi(v), \ M_2(v) \prec \Psi(v). \qquad (15.22)$$

PROOF. In virtue of Theorem 3.1, condition a) implies the condition of Lemma 15.1, condition b) implies the condition of Lemma 15.2, and condition c) implies the condition of Theorem 15.3. It is asserted in these lemmas and Theorem 15.3 that the conditions of Theorem 15.1 are satisfied and the proposition to be proved follows from the latter theorem. ★

We note that the N-function $\Psi(v)$ appearing in condition c) must satisfy the condition

$$|v|^{\alpha} \prec \Psi(v), \tag{15.23}$$

where $\alpha > 1$. This follows from the obvious fact that condition (15.23) is satisfied by every N-function whose complementary function satisfies the Δ_2-condition.

Conditions a), b), c) of Theorem 15.4 are not equivalent. Therefore, e.g., in the choice of the function $\Psi(v)$ satisfying either condition (15.20) or (15.21), it is natural to make clear at the start which of the compositions $M_2[N_1(v)]$ and $N_1[M_2(v)]$ increases "the slower." The conjecture arises that to answer this question it is sufficient to know which of the two relations $N_1(u) \prec M_2(u)$ and $M_2(u) \prec N_1(u)$ holds. But it turns out that this is not so.

For example, for $N_1(u) = u^2$, $M_2(u) = e^{|u|} - |u| - 1$ we have that the relation $N_1(u) \prec M_2(u)$ holds. In this connection, the N-functions $N_1[M_2(v)]$ and $M_2[N_1(v)]$ are not equivalent and $N_1[M_2(v)] \prec M_2[N_1(v)]$ inasmuch as $N_1[M_2(v)] \sim M_2(v)$, $M_2[N_1(v)] \sim e^{v^2} - 1$ and, for arbitrary $k > 0$, we have that

$$\lim_{v \to \infty} \frac{N_1[M_2(kv)]}{M_2[N_1(v)]} = 0.$$

The relation $N_1(u) \prec M_2(u)$ is also true for the N-functions $N_1(u) = e^{|u|} - |u| - 1$ and $M_2(u) = e^{u^2} - 1$; the N-functions $N_1[M_2(v)]$ and $M_2[N_1(v)]$ are again not equivalent — however, $M_2[N_1(v)] \prec N_1[M_2(v)]$ since, for arbitrary $k > 0$, we have that

$$\lim_{v \to \infty} \frac{N_1[M_2(v)]}{M_2[N_1(kv)]} = \infty.$$

A detailed comparison of conditions a), b) and c) of Theorem 15.4 will be executed below.

6. *On splitting a continuous operator.* Let $\boldsymbol{A}$ be a positive definite self-adjoint linear operator acting in the space L^2 of functions which are square-summable on G. As is known (see AHIEZER and GLAZMAN [1]), the operator $\boldsymbol{A}$ admits of the spectral decomposition

$$\boldsymbol{A} = \int_{\bullet}^{\infty} \lambda dE_{\lambda}$$

where E_λ is the spectral function of the operator A. The operator A^2 then admits of the spectral decomposition

$$A^2 = \int_0^\infty \lambda^2 dE_\lambda.$$

In the case when the operator A is completely continuous, the spectral decomposition is replaced by the infinite series

$$A\varphi(x) = \sum_{i=1}^{\infty} \lambda_i (e_i, \varphi) e_i(x) \quad (\varphi(x) \in L^2), \tag{15.24}$$

where the $e_i(x)$ are the characteristic functions of the operator A corresponding to the non-zero characteristic numbers λ_i. We denote the scalar product of the functions $e(x)$ and $\varphi(x)$ by

$$(e, \varphi) = \int_G e(x)\varphi(x)\, dx.$$

In this case,

$$A^2\varphi(x) = \sum_{i=1}^{\infty} \lambda_i^2 (e_{,i}\, \varphi) e_i(x).$$

We say that the operator B, acting from the Banach space E_1 into the Banach space E_2 with domain of definition $D(B)$, admits of a continuous extension $\bar{B}$, if $\bar{B} \in \{E_1 \to E_2; \text{c.}\}$ and $\bar{B}\varphi = B\varphi$ for $\varphi \in D(B)$.

THEOREM 15.5. *Let $M(u)$ and $N(u)$ be mutually complementary N-functions with $N(u) \prec u^2 \prec M(u)$. Suppose the operator A^2 (where A is a positive definite self-adjoint linear operator from $\{L^2 \to L^2; \text{c.}\}$) admits of a continuous extension to the operator $\bar{A}^2 \in \{E_N \to L_M^*; \text{c.}\}$. Then $\bar{A} \in \{L^2 \to L_M^*; \text{c.}\}$.*

PROOF. Let $\varphi(x) \in L^2$. Then, by Hölder's inequality,

$$\|A\varphi\|_{L^2}^2 = (A^2\varphi, \varphi) = (\bar{A}^2\varphi, \varphi) \leqslant \|\bar{A}^2\varphi\|_M \|\varphi\|_N \leqslant \|\bar{A}^2\| \|\varphi\|_N^2,$$

i.e.

$$\|A\varphi\|_{L^2} \leqslant k\|\varphi\|_N \quad (\varphi(x) \in L^2). \tag{15.25}$$

L^2 is dense in E_N since it contains all bounded functions. It therefore follows from (15.25) that the operator A admits of a continuous extension to all of E_N. The extended operator will be

denoted by A_1. The scalar product $l(\varphi) = (A_1\varphi, \psi)$, where $\psi(x)$ is a fixed element in L^2, defines a continuous linear functional on E_N. According to Theorem 14.2 on the general form of a linear functional on E_N, a function $u(x) \in L^*_M$ can be found such that

$$l(\varphi) = (\varphi, u) = \int_G \varphi(x)u(x)\,dx.$$

We define the operator A_1^* by means of the equality

$$A_1^*\psi(x) = u(x).$$

In virtue of (10.4),

$$\|A_1^*\psi\|_M = \sup_{\substack{\rho(\varphi;\,N)\leqslant 1\\ \varphi(x)\in E_N}} |(A_1^*\psi, \varphi)| \leqslant \sup_{\substack{\|\varphi\|_N\leqslant 2\\ \varphi(x)\in E_N}} |(\psi, A_1\varphi)| \leqslant 2k\|\psi\|_{L^2}.$$

Consequently, the operator A_1^* belongs to $\{L^2 \to L^*_M;\ \text{c.}\}$. We shall show that $A_1^*\psi(x) = A\psi(x)$ if $\psi(x) \in L^2$. This follows from the fact that $(A_1^*\psi, \varphi) = (\psi, A_1\varphi) = (\psi, A\varphi) = (A\psi, \varphi)$ for arbitrary function $\varphi(x) \in L^2$. ★

The square root $A^{\frac{1}{2}}$ of the operator A is defined as the operator whose square equals A. The operator $A^{\frac{1}{2}}$ has the spectral decomposition

$$A^{\frac{1}{2}} = \int_0^\infty \lambda^{\frac{1}{2}}\,dE_\lambda,$$

and, in the case of the completely continuous operator (15.24), we have that

$$A^{\frac{1}{2}}\varphi(x) = \sum_{i=1}^{\infty} \lambda_i^{\frac{1}{2}}(e_i, \varphi)e_i(x).$$

THEOREM 15.6. *Suppose the positive definite self-adjoint operator A, which is continuous in L^2, admits of a continuous extension to the operator $\bar{A} \in \{E_N \to L^*_M;\ \text{c.}\}$, where $N(u) \prec u^2 \prec M(u)$. Then the operator $\bar{A}$ admits of the representation*

$$\bar{A} = HH^*, \tag{15.26}$$

where $H \in \{L^2 \to L^*_M;\ \text{c.}\}$ and H^* is the operator in $\{E_N \to L^2;\ \text{c.}\}$ which is adjoint to H. [The operator H^* is defined with the aid of the equality $(H^*\varphi, \psi) = (\varphi, H\psi)$ $(\psi(x) \in L^2, \varphi(x) \in E_N)$.]

PROOF. We set $\boldsymbol{H} = \boldsymbol{A}^{\frac{1}{2}}$. In virtue of Theorem 15.5, $\boldsymbol{H}$ belongs to $\{L^2 \to L^*_M; \text{c.}\}$. The operator $\boldsymbol{HH}^*$ assumes the same values on L^2 as does the operator $\boldsymbol{A}$ inasmuch as $(\boldsymbol{HH}^*\varphi, \psi) = (\boldsymbol{H}^*\varphi, \boldsymbol{H}^*\psi) = (\boldsymbol{A}^{\frac{1}{2}}\varphi, \boldsymbol{A}^{\frac{1}{2}}\psi) = (\boldsymbol{A}\varphi, \psi)$ for an arbitrary pair of functions $\varphi(x)$, $\psi(x) \in L^2$. Therefore, the operator $\boldsymbol{HH}^*$ is a continuous extension of the operator $\boldsymbol{A}$ to an operator in $\{E_N \to L^*_M; \text{c.}\}$.

Equality (15.26) follows from the fact that a continuous extension of the operator $\boldsymbol{A}$ is unique since L^2 is dense in E_N with respect to the norm of the space E_N. ★

The representation $\bar{\boldsymbol{A}} = \boldsymbol{HH}^*$ will be called the *splitting of the operator* $\boldsymbol{A}$.

§ 16. Conditions for the complete continuity of linear integral operators

1. *The case of continuous kernels.* We now continue with the study of the linear integral operator

$$\boldsymbol{A}u(x) = \int_G k(x, y)u(y)\,dy. \tag{16.1}$$

In the present section, we shall study the problem of the conditions for the complete continuity of the operator (16.1), i.e. the conditions under which the operator (16.1) maps the unit sphere of the space $L^*_{M_1}$ into a compact set in the space $L^*_{M_2}$.

LEMMA 16.1. *Suppose the kernel $k(x, y)$ is continuous on $\hat{G}$. Let $L^*_{M_1}$ and $L^*_{M_2}$ be two arbitrary Orlicz spaces. Then the operator* (16.1) *belongs to* $\{L^*_{M_1} \to E_{M_2}; \text{ comp. c.}\}$.

PROOF. In virtue of Theorem 15.4, the operator $\boldsymbol{A}$ belongs to $\{L^*_{M_1} \to L^*_{M_2}; \text{c.}\}$.

Let T be the unit sphere of the space $L^*_{M_1}$. Since

$$\int_G |u(x)|\,dx \leqslant \|u\|_{M_1}\|\kappa(x; G)\|_{N_1} \leqslant \operatorname{mes} G\, M_1^{-1}\left(\frac{1}{\operatorname{mes} G}\right)$$

for $u(x) \in T$, we have, for $x \in G$, $u(x) \in T$, that

$$|\boldsymbol{A}u(x)| = \left|\int_G k(x, y)u(y)\,dy\right| \leqslant K \operatorname{mes} G\, M_1^{-1}\left(\frac{1}{\operatorname{mes} G}\right),$$

where $K = \max |k(x, y)|$ $(x, y \in G)$.

This means that the functions $|Au(x)|$ $(u(x) \in T)$ are uniformly bounded.

Let $\varepsilon > 0$ be prescribed. We choose a $\delta > 0$ such that $|k(x_1, y) - k(x_2, y)| < \varepsilon/[\text{mes } G M_1^{-1}(1/\text{mes } G)]$ for $d(x_1, x_2) < \delta$, where $d(x_1, x_2)$ denotes the distance between the points $x_1, x_2 \in G$. Then, with $d(x_1, x_2) < \delta$, for an arbitrary function $u(x) \in T$, we have, in virtue of the formula for the norm of the characteristic function of the set G in the space $L^*_{N_1}$, that

$$|Au(x_1) - Au(x_2)| \leqslant \int_G |k(x_1, y) - k(x_2, y)|\,|u(y)|\,dy \leqslant$$

$$\leqslant \frac{\varepsilon}{\text{mes } G M_1^{-1}\left(\dfrac{1}{\text{mes } G}\right)} \int_G |u(y)|\,dy \leqslant \varepsilon.$$

The functions $Au(x)$ $(u(x) \in T)$ are thus equi-continuous.

In virtue of Arzelà's theorem, the set AT is compact in the space C of functions which are continuous on G and, a fortiori, compact on an arbitrary Orlicz space.

Since the functions $Au(x)$ are continuous, they belong to E_{M_2}. ★

2. *Fundamental theorem.* Conditions for the complete continuity of operators of type (16.1) can be obtained by making use of criteria for the compactness of a family of functions in Orlicz spaces. A simpler way consists in establishing the possibility of an arbitrarily close approximation of the operator (16.1) by a known completely continuous operator. It is convenient to consider, as such approximating operators, also integral operators but with continuous kernels.

In some cases, the conditions of Theorems 15.1 and 15.4 are sufficient for the complete continuity of the operator (16.1). But it is unknown if they are sufficient in the general case. It turns out that the complete continuity of operator (16.1) will be guaranteed if the condition $k(x, y) \in \hat{L}^*_\Psi$ in Theorems 15.1 and 15.4 is replaced by the more severe condition $k(x, y) \in \hat{E}_\Psi$. We shall prove this fact, which will be utilized in the sequel, from the corresponding two assertions.

THEOREM 16.1 (CONCERNING SUFFICIENT CONDITIONS FOR COMPLETE CONTINUITY). *Let $\Phi(u)$ and $\Psi(v)$ be mutually complementary N-functions. Suppose the kernel $k(x, y)$ of the linear integral operator*

(16.1) *belongs to the space* $\hat{E}_\Psi$. *Then each of the following conditions* a), b), c) *of Theorem* 15.4 *is sufficient for the operator* (16.1) *to belong to* $\{L^*_{M_1} \to E_{M_2}$; comp. c.$\}$:

a) $M_2[N_1(v)] \prec \Psi(v)$;

b) $N_1[M_2(v)] \prec \Psi(v)$;

c) *the function* $\Phi(u)$ *satisfies the* Δ'*-condition and* $N_1(v) \prec \Psi(v)$, $M_2(v) \prec \Psi(v)$.

PROOF. Since $k(x, y) \in \hat{E}_\Psi$, a sequence $k_n(x, y)$ $(n = 1, 2, \cdots)$ of continuous kernels can be constructed such that

$$\|k(x, y) - k_n(x, y)\|_\Psi < 1/n.$$

We shall denote the linear integral operators

$$\boldsymbol{A}_n u(x) = \int_G k_n(x, y)u(y)\,dy$$

by $\boldsymbol{A}_n$. In virtue of Lemma 16.1, these operators act from $L^*_{M_1}$ into E_{M_2} and they are completely continuous.

Under the conditions of the theorem just proved, the conditions of Theorem 15.1 are also satisfied. Therefore, in virtue of (15.6), we have that

$$\|\boldsymbol{A} - \boldsymbol{A}_n\| \leqslant 2l\|k(x, y) - k_n(x, y)\|_\Psi < 2l/n \quad (n = 1, 2, \cdots).$$

This means that the operator $\boldsymbol{A}$ can be approximated arbitrarily closely in norm by a completely continuous operator with values in E_{M_2}. And this implies the assertion of the theorem.

Theorem 16.1 can be applied in two different variants.

First, the problem can be posed on the properties of the function $\Psi(v)$ under which the operator (16.1) acts from the *given* space $L^*_{M_1}$ into the given space E_{M_2} and is completely continuous. In this case, the application of Theorem 16.1 depends on the verification of the fact whether the kernel $k(x, y)$ belongs to the space $\hat{E}_\Psi$, i.e. is the condition

$$\int_G\int \Psi[\lambda k(x, y)]\,dx\,dy < \infty \tag{16.2}$$

satisfied for all λ?

If the N-function $\Psi(v)$ satisfies the Δ_2-condition, then (16.2) is equivalent to the condition

$$\int\int_G \Psi[k(x, y)]\,dx\,dy < \infty. \tag{16.3}$$

But if the N-function $\Psi(v)$ does not satisfy the Δ_2-condition, then the verification of condition (16.2) becomes rather difficult. It is easily seen that (16.2) is satisfied if

$$\int\int_G \Psi\{Q[k(x, y)]\}\,dx\,dy < \infty, \tag{16.4}$$

where $Q(u)$ is an N-function. We also note that, under the conditions (16.2), (16.3) and (16.4), one can use, instead of the function $\Psi(v)$, any other function $R(v)$ which is the principal part of an N-function which is equivalent to $\Psi(v)$.

Second, the problem can be posed concerning finding N-functions $M_1(u)$ and $M_2(u)$ such that the operator (16.1) with given kernel $k(x, y)$ acts from $L^*_{M_1}$ into E_{M_2} and is completely continuous. The N-functions $M_1(u)$ and $M_2(u)$ are then chosen so that one of the conditions (15.20), (15.21) and (15.22) is satisfied.

3. *Complete continuity and E_N-weak convergence.* As is known, completely continuous linear operators acting from one Banach space into another transform a weakly convergent sequence of elements into a sequence which converges in norm.

Since the class of E_N-weakly convergent sequences in an Orlicz space is, generally speaking, more extensive than the class of sequences which converge weakly in the usual sense, supplementary conditions are required in order that a completely continuous linear operator acting from one Orlicz space $L^*_{M_1}$ into another Orlicz space $L^*_{M_2}$ possess the analogous property with respect to E_N-weak convergence.

Here, we shall elucidate this problem for the linear integral operator (16.1). In this connection, we shall assume that the kernel $k(x, y)$ of the operator $\boldsymbol{A}$ satisfies the following condition:

$$\int\int_G k(x, y)u(y)v(x)\,dx\,dy < \infty \tag{16.5}$$

for an arbitrary pair of functions $u(x) \in L^*_{M_1}$, $v(x) \in L^*_{N_2}$. In virtue of this condition, the operator A acts from $L^*_{M_1}$ into $L^*_{M_2}$ and the operator A^*, defined by the equality

$$A^*v(x) = \int_G k(y, x)v(y)\,dy, \tag{16.6}$$

acts from $L^*_{N_2}$ into $L^*_{N_1}$, in which connection, the inequality

$$\int_G Au(x)v(x)\,dx = \int_G u(y)A^*v(y)\,dy \tag{16.7}$$

holds for an arbitrary pair of functions $u(x) \in L^*_{M_1}$, $v(x) \in L^*_{N_2}$.

LEMMA 16.2. *A necessary and sufficient condition that the continuous linear integral operator A with kernel $k(x, y)$, satisfying condition* (16.5), *transform every E_{N_1}-weakly convergent sequence of functions in $L^*_{M_1}$ into an E_{N_2}-weakly convergent sequence of functions in $L^*_{M_2}$ is that the operator A^* act from E_{N_2} into E_{N_1}.*

PROOF. The sufficiency of the condition of the lemma is evident inasmuch as, in virtue of equality (16.7), the sequence of numbers

$$l(Au_n) = \int_G A_n u(x)v(x)\,dx$$

converges for arbitrary function $v(x) \in E_{N_2}$ in virtue of the fact that $A^*v(x) \in E_{N_1}$.

We shall now prove the necessity of the condition of the lemma. Suppose the continuous linear integral operator A transforms every E_{N_1}-weakly convergent sequence of functions in $L^*_{M_1}$ into an E_{N_2}-weakly convergent sequence of functions in $L^*_{M_2}$. Let $v(x) \in E_{N_2}$. Then the functional $l(u)$, defined on $L^*_{M_1}$ by means of the equality

$$l(u) = \int_G Au(x)v(x)\,dx,$$

will be an E_{N_1}-weakly continuous functional on $L^*_{M_1}$. In virtue of Theorem 14.7, this functional is representable in the form

$$l(u) = \int_G u(x)v^*(x)\,dx,$$

where $v^*(x) \in E_{N_1}$.

Since, on the other hand, in virtue of (16.7),

$$\int_G u(x)v^*(x)\,dx = \int_G u(x)\boldsymbol{A}^*v(x)\,dx$$

for arbitrary function $u(x) \in L^*_{M_1}$, we have that $v^*(x) = \boldsymbol{A}^*v(x)$ for almost all $x \in G$, i.e. that $\boldsymbol{A}^*v(x) \in E_{N_1}$. ★

The following theorem can be easily obtained from this lemma.

THEOREM 16.2. *A necessary and sufficient condition that the completely continuous linear integral operator* $\boldsymbol{A}$ *with kernel* $k(x, y)$, *satisfying condition* (16.5), *transform every* E_{N_1}*-weakly convergent sequence of functions in* $L^*_{M_1}$ *into a sequence of functions which converges with respect to the norm in* $L^*_{M_2}$ *is that the operator* $\boldsymbol{A}^*$ *act from* E_{N_2} *into* E_{N_1}.

PROOF. The necessity of the condition of the theorem follows, in an obvious manner, from Lemma 16.2. The sufficiency follows from Lemma 16.2, Theorem 14.3 and Theorem 14.5. ★

We note that, under the conditions of Theorem 16.1 on the complete continuity of the operator (16.1), condition (16.5) is satisfied and the operator $\boldsymbol{A}^*$, defined by equality (16.6), acts from $L^*_{N_2}$ into E_{N_1} (and, a fortiori, from E_{N_2} into E_{N_1}). We shall prove the last assertion.

Under the conditions of Theorem 16.1, conditions (15.2) and (15.3) are satisfied. Suppose $v(x) \in L^*_{N_2}$ and let ε be an arbitrary positive number. Since the kernel $k(x, y)$, belonging to $\hat{E}_\Psi$, has an absolutely continuous norm in $\hat{L}^*_\Psi$, a $\delta > 0$ can be found such that mes $\mathscr{E} < \delta$ implies that $\|k(x, y)\kappa(x, y; \hat{\mathscr{E}})\|_\Psi < \varepsilon/\{2\|v\|_{N_2}l\}$ $(\hat{\mathscr{E}} \subset \hat{G})$, where l is the constant in condition (15.3). Let $\mathscr{E} \subset G$, $u(x) \in L_{M_1}$. Then

$$\int_{\mathscr{E}} \boldsymbol{A}^*v(x)u(x)\,dx = \int_{\mathscr{E}}\int_G k(y, x)v(y)u(x)\,dx\,dy =$$

$$= \iint_{\hat{\mathscr{E}}} k(y, x)v(y)u(x)\,dx\,dy,$$

where $\hat{\mathscr{E}} = \mathscr{E} \times G$.

Now let mes $\mathscr{E} < \delta/\text{mes } G$ and $\rho(u; M) \leqslant 1$. Then mes $\hat{\mathscr{E}} =$ $=$ mes $\mathscr{E}$ mes $G < \delta$, $\|u\|_M \leqslant 1 + \rho(u, M) \leqslant 2$. Applying the Höl-

der inequality and (15.3) to the last integral, we obtain that

$$\left| \int_{\mathscr{E}} \boldsymbol{A}^* v(x) u(x)\, dx \right| \leqslant l \|k(x, y)\kappa(x, y; \hat{\mathscr{E}})\|_{\Psi} \cdot 2\|v\|_{N_2} < \varepsilon.$$

It follows from this that $\|\boldsymbol{A}^* v(x)\kappa(x; \mathscr{E})\|_{N_1} < \varepsilon$ provided mes $\mathscr{E} < \delta/\text{mes } G$. Thus, the function $\boldsymbol{A}^* v(x)$ has in $L^*_{N_1}$ an absolutely continuous norm and, consequently, it belongs to E_{N_1}.

Thus, under the conditions of Theorem 16.1 — without any other, supplementary, assumptions — the operator (16.1) transforms every E_{N_1}-weakly convergent sequence of functions into a sequence of functions which converges with respect to the norm in $L^*_{M_2}$.

We note further that the assertion of Theorem 16.2 remains valid for abstract linear completely continuous operators $\boldsymbol{A}$ if the N-function $M_1(u)$ satisfies the Δ_2-condition and the operator $\boldsymbol{A}^*$ is defined in the following way. Let $v(x) \in L^*_{N_2}$. We consider the linear functional, defined, on $L^*_{M_1} = E_{M_1}$, by means of the equality

$$\boldsymbol{l}(u) = \int_G \boldsymbol{A}u(x) v(x)\, dx.$$

In virtue of Theorem 14.2, this functional can be represented in the form

$$\boldsymbol{l}(u) = \int_G u(x) v^*(x) dx,$$

where $v^*(x) \in L^*_{N_1}$. We set $\boldsymbol{A}^* v(x) = v^*(x)$. Thus, the operator $\boldsymbol{A}^*$ acts from $L^*_{N_2}$ into $L^*_{N_1}$ and (16.7) holds.

4. *Zaanen's theorem.*

LEMMA 16.3. *Suppose the kernel $k(x, y)$ satisfies either one of the following two conditions*:

a) *for almost all $x \in G$, the kernel $k(x, y)$, as a function of y, belongs to the space $L^*_{N_1}$, where the function $\varphi(x) = \|k(x, y)\|_{N_1}$ belongs to the space $L^*_{M_2}$;*

b) *for almost all $y \in G$, the kernel, as a function of x, belongs to the space $L^*_{M_2}$, where the function $\Psi(y) = \|k(x, y)\|_{M_2}$ belongs to the space $L^*_{N_1}$.*

Then the operator (16.1) *belongs to* $\{L^*_{M_1} \to L^*_{M_2}; \text{c.}\}$.

PROOF. Suppose the condition a) is satisfied. Then, in virtue

of Hölder's inequality,

$$\left| \int_G Au(x)v(x)\,dx \right| \leqslant$$

$$\leqslant \int_G \left| \int_G k(x, y)u(y)\,dy \right| |v(x)|\,dx \leqslant \|u\|_{M_1} \int_G \varphi(x)\,|v(x)|\,dx$$

for arbitrary functions $u(x) \in L^*_{M_1}$, $v(x) \in L_{N_2}$, from which it follows that

$$\|Au\|_{M_2} = \sup_{\rho(v;\,N_2)\leqslant 1} \left| \int_G Au(x)v(x)\,dx \right| \leqslant \|\varphi\|_{N_2}\|u\|_{M_1}.$$

If condition b) is satisfied, then, interchanging the order of integration (which is possible on the basis of Fubini's theorem), we obtain that

$$\left| \int_G Au(x)v(x)\,dx \right| \leqslant \int_G \left| \int_G k(x, y)v(x)\,dx \right| |u(y)|\,dy \leqslant$$

$$\leqslant \|v\|_{N_2} \int_G \psi(y)\,|u(y)|\,dy \leqslant \|v\|_{N_2}\|\psi\|_{N_1}\|u\|_{M_1}$$

for arbitrary functions $u(x) \in L^*_{M_1}$, $v(x) \in L_{N_2}$, from which it follows that

$$\|Au\|_{M_2} = \sup_{\rho(v;\,N_2)\leqslant 1} \left| \int_G Au(x)v(x)\,dx \right| \leqslant$$

$$\leqslant \sup_{\|v\|_{N_2}\leqslant 2} \left| \int_G Au(x)v(x)\,dx \right| \leqslant 2\|\psi\|_{N_1}\|u\|_{M_1}. \bigstar$$

THEOREM 16.3. *Suppose the kernel $k(x, y)$, as a function of y, belongs to E_{N_1} for almost all $x \in G$, where the function $\varphi(x) = \|k(x, y)\|_{N_1}$ belongs to E_{M_2}. Then the operator* (16.1) *belongs to* $\{L^*_{M_1} \to E_{M_2}$; comp. c.$\}$.

PROOF. In virtue of the preceding lemma, the operator (16.1) acts from $L^*_{M_1}$ into $L^*_{M_2}$ and is continuous. It remains to show that this operator transforms the unit sphere T of the space $L^*_{M_1}$ into a set of functions which is compact in E_{M_2}. In virtue of the E_{N_1}-weak compactness of the sphere T, it suffices to show that the operator A transforms an E_{N_1}-weakly convergent sequence of

functions on the sphere T into a sequence of functions in E_{M_2} which converges in norm.

Suppose the sequence $u_n(x) \in T$ $(n = 1, 2, \cdots)$ is E_{N_1}-weakly convergent to the function $u_0(x)$. In virtue of the condition of the theorem, the sequence of functions $Au_n(x)$ $(n = 1, 2, \cdots)$ converges to the function $Au_0(x)$ almost everywhere and, consequently, it converges to this function in measure. We shall show that the functions $Au_n(x)$ have equi-absolutely continuous norms in $L^*_{M_2}$. Let $\kappa(x; \mathscr{E})$ be the characteristic function of the set $\mathscr{E} \subset G$, $v(x) \in L_{N_2}$. Then

$$\left| \int_{\mathscr{E}} Au_n(x)v(x)\,dx \right| \leqslant$$

$$\leqslant \int_{\mathscr{E}} \left| \int_G k(x, y)u_n(y)\,dy \right| |v(x)|\,dx \leqslant \int_{\mathscr{E}} \varphi(x)\,|v(x)|\,dx,$$

from which it follows that

$$\|Au_n(x)\kappa(x; \mathscr{E})\|_{M_2} = \sup_{\rho(v;\,N_2)\leqslant 1} \left| \int_{\mathscr{E}} Au_n(x)v(x)\,dx \right| \leqslant \|\varphi(x)\kappa(x; \mathscr{E})\|_{M_2}.$$

Since the function $\varphi(x)$ belongs to E_{M_2}, it follows that the functions $Au_n(x)$ have equi-absolutely continuous norms.

In virtue of Lemma 11.2, the sequence $Au_n(x)$, which belongs to E_{M_2}, converges in norm. ★

Theorem 16.3 can be augmented by the following assertion.

THEOREM 16.4. *Suppose the kernel $k(x, y)$, as a function of x, belongs to E_{M_2} for almost all $y \in G$, where the function $\psi(y) = \|k(x, y)\|_{M_2}$ belongs to E_{N_1}. Then the operator* (16.1) *belongs to* $\{E_{M_1} \to E_{M_2};$ comp. c.$\}$.

PROOF. In virtue of Lemma 16.3, $A \in \{L^*_{M_1} \to L^*_{M_2};$ c.$\}$. We consider the operator A^* defined by means of equality (16.6). In virtue of Theorem 16.3, the operator A^* belongs to $\{L^*_{N_2} \to E_{N_1};$ comp. c.$\}$ and transforms every E_{M_2}-weakly convergent sequence of functions in $L^*_{N_2}$ into a sequence of functions in E_{N_1} which converges in norm. In virtue of Theorem 16.2, the operator $A = (A^*)^*$ acts from E_{M_1} into E_{M_2}. In order to complete the proof, it remains to prove that the operator A transforms every E_{N_1}-weakly convergent sequence of functions from the unit sphere

T of the space $L^*_{M_1}$ into a sequence which converges with respect to the norm in $L^*_{M_2}$.

Suppose the sequence of functions $u_n(x) \in T$ $(n = 1, 2, \cdots)$ is E_{N_2}-weakly convergent to the function $u_0(x)$. It is easily seen that $\|u_0\|_{M_1} \leqslant 2$. The sequence $Au_n(x)$ is E_{N_2}-weakly convergent to the function $Au_0(x)$ inasmuch as for every function $u(x) \in E_{N_2}$ we have, in virtue of equality (16.7), that

$$\lim_{n\to\infty} \int_G A[u_n(x) - u_0(x)]v(x)\,dx = \lim_{n\to\infty} \int_G [u_n(x) - u_0(x)]A^*v(x)\,dx = 0.$$

Let $\varepsilon > 0$ be given. We denote by $A^*v_1(x)$, $A^*v_2(x)$, $\cdots$, $A^*v_s(x)$ a finite $(\varepsilon/6)$-net of the set $\{A^*v\}$ $(\rho(v; N_2) \leqslant 1)$, which is compact in virtue of the complete continuity of the operator A^*. Then, for every function $v(x) \in L_{N_2}$ $(\rho(v; N_2) \leqslant 1)$ we can find a function $v_{i(v)}(x)$ such that $\|A^*v - A^*v_{i(v)}\|_{N_1} < \varepsilon/6$.

Suppose the inequalities

$$\int_G [u_n(x) - u_0(x)]A^*v_i(x)\,dx < \frac{\varepsilon}{2} \quad (i = 1, 2, \cdots, s)$$

are satisfied for $n \geqslant n_0$. Then, for $n \geqslant n_0$, we have that

$$\|Au_n - Au_0\|_{M_2} = \sup_{\rho(v; N_2)\leqslant 1} \left| \int_G A[u_n(x) - u_0(x)]v(x)\,dx \right| =$$

$$= \sup_{\rho(v; N_2)\leqslant 1} \left| \int_G [u_n(x) - u_0(x)]A^*v(x)\,dx \right| \leqslant$$

$$\leqslant \sup_{\rho(v; N_2)\leqslant 1} \left| \int_G [u_n(x) - u_0(x)]A^*v_{i(v)}(x)\,dx \right| +$$

$$+ \sup_{\rho(v; N_2)\leqslant 1} \int_G |u_n(x) - u_0(x)\|A^*v(x) - A^*v_{i(v)}(x)|\,dx <$$

$$< \frac{\varepsilon}{2} + \|u_n - u_0\|_{M_1} \sup_{\rho(v; N_2)\leqslant 1} \|A^*v_{i(v)} - A^*v\|_{N_1} < \varepsilon,$$

i.e. the sequence $Au_n(x)$ converges in norm to $Au_0(x)$. ★

We note that the sufficiency of condition a) of Theorem 16.1 can also be obtained as a consequence of Theorem 16.3. In fact, suppose the kernel $k(x, y)$ belongs to the space $\hat{E}_{M_2[N_1]}$. Then, for arbitrary $\lambda > 0$, we have that

$$\int_G \int M_2\{N_1[\lambda k(x, y)]\}\, dx\, dy < \infty. \tag{16.8}$$

It follows from this that the conditions of Theorem 16.3 are satisfied. In fact, it follows from (16.8) that

$$\int_G N_1[\lambda k(x, y)]\, dx < \infty$$

for arbitrary $\lambda > 0$ and for almost all $x \in G$. This means that for almost all $x \in G$ the kernel $k(x, y)$, as a function of y, belongs to the space E_{N_1}. In virtue of (9.12), the estimate

$$\varphi(x) \leqslant 1 + \int_G N_1[k(x, y)]\, dy$$

is valid for the function $\varphi(x) = \|k(x, y)\|_{N_1}$.

In order to complete the proof of the theorem, it suffices to show that

$$\int_G M_2[\mu\varphi(x)]\, dx < \infty$$

for arbitrary $\mu > (2 \operatorname{mes} G)^{-1}$. This fact follows from (16.8) inasmuch as

$$\int_G M_2[\mu\varphi(x)]\, dx \leqslant \int_G M_2\left\{\mu + \mu \int_G N_1[k(x, y)]\, dy\right\} dx \leqslant$$

$$\leqslant \tfrac{1}{2} M_2(2\mu) \operatorname{mes} G + \tfrac{1}{2} \int_G M_2\left\{\frac{\int_G N_1[2\mu \operatorname{mes} G k(x, y)]\, dx}{\operatorname{mes} G}\right\} dx \leqslant$$

$$\leqslant \tfrac{1}{2} M_2(2\mu) \operatorname{mes} G +$$

$$+ \frac{1}{2 \operatorname{mes} G} \int_G \int M_2\{N_1[2\mu \operatorname{mes} G k(x, y)]\}\, dx\, dy < \infty.$$

5. *Comparison of conditions.* As a first example, we shall consider the simplest case — when $L^*_{M_1} = L^*_{M_2} = L^2$. In this case, conditions a) and b) of Theorem 16.1 signify that the kernel $k(x, y)$ must satisfy the relation

$$\int_G \int k^4(x, y)\, dx\, dy < \infty.$$

But if we make use of condition c), taking the function v^2 for the function $\Psi(v)$, it turns out that the satisfaction of the relation

$$\int_G \int k^2(x, y)\, dx\, dy < \infty \tag{16.9}$$

is sufficient for the complete continuity of the operator (16.1) in L^2. Condition (16.9) is less restrictive.

We note that the known condition (16.9) is not necessary for the complete continuity of the linear integral operator in L^2. It is clear that all the conditions, indicated above, for complete continuity in the case of arbitrary Orlicz spaces are only sufficient.

As a second example, we note that it follows from this same condition c) of Theorem 16.1 that the linear integral operator (16.1) acts from L^{α_1} into L^{α_2} ($\alpha_1 > 1$, $\alpha_2 > 1$) and is completely continuous if the kernel $k(x, y)$ is p-*th* power summable on G with p equal to $\max\{\beta_1, \alpha_2\}$, where $1/\alpha_1 + 1/\beta_1 = 1$. In order to obtain this known result (which, of course, is proved directly quite simply), it suffices to set $\Psi(v) = |v|^{\max\{\beta_1, \alpha_2\}}$.

Now, let $M_1(u) = (1 + |u|) \ln (1 + |u|) - |u|$, $M_2(u) = e^{|u|} - |u| - 1$. In this case, $N_1(u) = M_2(u)$. From Theorem 16.1 there follow three conditions which are sufficient that the operator (16.1) act from $L^*_{M_1} = L_M$ into E_{M_2} and be completely continuous. These conditions can be written in the form

$$\int_G \int \exp\{\exp|\lambda k(x, y)|\}\, dx\, dy < \infty$$

if we make use of conditions a) or b), and in the form

$$\int_G \int \exp|\lambda k(x, y)|\, dx\, dy < \infty \tag{16.10}$$

if we make use of condition c), setting $\Psi(v) = M_2(v)$.

Thus, in this case also, condition c) turns out to be less restrictive.

The last example is an illustration of the following theorem (which follows directly from Theorem 16.1) — this theorem will play an important role in the study of integral equations with essentially non-power non-linearities.

THEOREM 16.5. *Suppose $M(u)$ and $N(v)$ are mutually complementary N-functions where $N(v)$ satisfies the Δ'-condition. Let*

$$\int\int_G M[\lambda k(x, y)]\,dx\,dy < \infty$$

for arbitrary $\lambda > 0$. Then the linear integral operator (16.1) *belongs to* $\{L_N \to E_M$; comp. c.$\}$.

In the next example, we set $M_1(u) = |u|^\alpha/\alpha$ $(\alpha > 1)$, $M_2(u) = e^{|u|} - |u| - 1$. For the complete continuity of operator (16.1), it is sufficient: in virtue of condition a) that

$$\int\int_G \exp|\lambda k(x, y)|^\beta\,dx\,dy < \infty \quad \left(\frac{1}{\alpha} + \frac{1}{\beta} = 1\right)$$

for arbitrary $\lambda > 0$, and in virtue of conditions b) or c) that

$$\int\int_G \exp|\lambda k(x, y)|\,dx\,dy < \infty.$$

The second condition is, of course, less restrictive. We note, further, that it coincides with condition (16.10), which was obtained in the preceding example. This is not accidental since the following more general assertion holds.

THEOREM 16.6. *Let*

$$M_1(u) \prec \Phi_1(u),\ \Phi_2(u) \prec M_2(u). \tag{16.11}$$

Then every linear operator in $\{L^*_{M_1} \to E_{M_2}$; comp. c.$\}$ *belongs to* $\{L^*_{\Phi_1} \to E_{\Phi_2}$; comp. c.$\}$.

PROOF. The fact that $A \in \{L^*_\Phi \to E_{\Phi_2}\}$ follows from the inclusions $L^*_{\Phi_1} \subset L^*_{M_1}$, $E_{M_2} \subset E_{\Phi_2}$. In virtue of Theorem 13.3 and (16.11), the inequalities $\|u\|_{M_1} \leqslant q_1\|u\|_{\Phi_1}$ $(u(x) \in L^*_{\Phi_1})$ and $\|u\|_{\Phi_2} \leqslant q_2\|u\|_{M_2}$ $(u(x) \in L^*_{M_2})$ hold. Let T be a bounded set in $L^*_{\Phi_1}$. In virtue of the first of the above inequalities, T is also bounded in $L^*_{M_1}$. Therefore, the set AT is compact in $L^*_{M_2}$. In virtue of the second inequality, it is also compact in $L^*_{\Phi_2}$. ★

It follows from the preceding examples that condition c) is less restrictive, in a number of cases, than conditions a) and b). We shall now give an example which shows that this is not always so.

Let $M_1(u) = e^{|u|} - |u| - 1$, $M_2(u) = (1 + |u|) \ln (1 + |u|) - |u|$. In this case, $E_{M_2} = L^*_{M_2} = L_{M_2}$. It follows from conditions a) and b) of Theorem 16.1 that the operator (16.1) belongs to $\{L^*_{M_1} \to E_{M_2}$; comp. c.$\}$ if the inequality

$$\int_G \int |k(x, y)| \ln^2 (1 + |k(x, y)|)\, dx\, dy < \infty \qquad (16.12)$$

is satisfied. Application of condition a) leads to the assumption that the kernel $k(x, y)$ is summable with some power $\alpha > 1$, i.e. it leads to a more restrictive condition (see Remark, p. 146).

Summarizing everything stated above, we can formulate a rule for finding a function $\Psi(v)$ satisfying the conditions of Theorem 16.1.

In this connection, we shall assume that any two N-functions $\Phi_1(u)$ and $\Phi_2(u)$, which may be under consideration, are "comparable," i.e. that one of the relations: $\Phi_1(u) \prec \Phi_2(u)$ and $\Phi_2(u) \prec \Phi_1(u)$ is satisfied. In the first case, we shall say that $\Phi_1(u)$ is "smaller" than $\Phi_2(u)$ and in the second that $\Phi_1(u)$ is "larger" than $\Phi_2(u)$.

Let the spaces $L^*_{M_1}$ and $L^*_{M_2}$ be given. We consider the N-functions $M_1(u)$ and $N_2(u)$. If the "smaller" of them satisfies the Δ'-condition, then we shall consider the complementary N-function to it to be equal to $\Psi(v)$. Obviously, in this case condition c) is satisfied — which leads to the least (in comparison with conditions a) and b)) restrictions on the kernel $k(x, y)$.

But if the "smaller" of the functions $M_1(u)$ and $N_2(u)$ does not satisfy the Δ'-condition then two cases are possible:

1. Both of the functions $M_1(u)$ and $N_2(u)$ increase more rapidly than an arbitrary power $|u|^\alpha$ $(\alpha > 1)$. We shall assume that they satisfy the Δ_3-condition. Then the functions $N_1(v)$ and $M_2(v)$ increase less rapidly than any power $|v|^\beta$ $(\beta > 1)$. Making use of condition a) or b) of Theorem 16.1 and Theorem 6.9, we can set $\Psi(v) = N_1(v)M_2(v)/|v|$ which is equivalent, in this case, to one of the functions $M_2[N_1(v)]$ or $N_1[M_2(v)]$ and is "smaller" than the other. It is easily seen that the function $\Psi(v)$ also increases less rapidly than any power of the form $|v|^\beta$ $(\beta > 1)$. But any function $\Psi(v)$, which satisfies condition c) in virtue of the

remark on p. 146, increases more rapidly than some power function $|v|^{\beta_0}$ ($\beta_0 > 1$), which leads to a greater restriction on the kernel $k(x, y)$.

2. The "smaller" of the functions $M_1(u)$ and $N_2(u)$ does not satisfy the Δ'-condition but it does not satisfy the Δ_3-condition either. In this case, we can make use of condition c), taking as the function $\Psi(v)$ some N-function which is larger than $N_1(v)$ and $M_2(v)$ but whose complementary function satisfies the Δ'-condition. One can also make use of conditions a) or b), taking as the function $\Psi(v)$ the function $M_2[N_1(v)]$ or $N_1[M_2(v)]$. In some cases, the first method leads to a lesser restriction with respect to the kernel $k(x, y)$ and, in other cases, the second. We shall elucidate this by means of examples.

Let $M_1(u) = u^2/2$, $M_2(u) = u^2(|\ln|u|| + 1)$. In this case, the function $N_2(u)$ is equivalent to the N-function $u^2/\ln(|u| + e) \prec \prec M_1(u)$ and does not satisfy the Δ'-condition. Making use of condition c), setting $\Psi(v) = |v|^{2+\varepsilon}$, where ε is a positive number, we arrive at the following restriction on the kernel $k(x, y)$:

$$\int\int_G |k(x, y)|^{2+\varepsilon} dx\, dy < \infty.$$

Application of condition a) or b) leads to the worse conditions:

$$\int\int_G |k(x, y)|^4 \ln^2(|k(x, y)| + 1)\, dx\, dy < \infty$$

or

$$\int\int_G |k(x, y)|^4 \ln(|k(x, y)| + 1)\, dx\, dy < \infty,$$

respectively.

Now let $M_1(u) = e^{|u|} - |u| - 1$, $M_2(u) = u^2(|\ln|u|| + 1)$. In this case, the N-function $N_2(u)$ is also "smaller" than $M_1(u)$ and does not satisfy the Δ'-condition. Application of condition a) leads to the following restriction on the kernel $k(x, y)$:

$$\int\int_G |k(x, y)|^2 \ln^3(|k(x, y)| + 1)\, dx\, dy < \infty$$

and application of condition b) leads to the restriction

$$\int\int_G |k(x, y)|^2 \ln^2 (|k(x, y)| + 1)\, dx\, dy < \infty.$$

But to apply condition c) one must take as the function $\Phi(u)$, complementary to $\Psi(v)$, a function which is "smaller" than $u^2/\ln(|u| + e)$ and which satisfies the Δ'-condition. Such a function can be, for example, the function $|u|^{2-\varepsilon}$, where $0 < \varepsilon < 1$. But then the function $\Psi(v)$ will increase as $|v|^{2+\varepsilon_1}$ $(\varepsilon_1 > 0)$ which leads to a worse condition.

6. *On splitting a completely continuous operator.*

THEOREM 16.7. *Let $M(u)$ and $N(u)$ be mutually complementary N-functions with $N(u) \prec u^2 \prec M(u)$. Suppose the operator A^2 (where A is a positive definite self-adjoint linear operator in $\{L^2 \to L^2$; comp. c.$\}$) admits of a continuous extension to the operator $\bar{A}^2 \in \{E_N \to L_M^*$; comp. c.$\}$. Then $A \in \{L^2 \to L_M^*$; comp. c.$\}$.*

PROOF. Since the conditions of Theorem 15.5 are satisfied, we have that $A \in \{L^2 \to L_M^*$; c.$\}$.

Let A_1 be the operator acting from E_N into L^2, introduced in the proof of Theorem 15.5. As was proved, A is the adjoint operator to A_1. Therefore, to prove the complete continuity of the operator A, it suffices to prove that the operator A_1 is completely continuous.

Let $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ be an E_N-weakly convergent sequence of functions in L^2. Then $A_1\varphi_i(x) = A\varphi_i(x)$ and

$$\|A_1(\varphi_n - \varphi_m)\|_{L^2}^2 = (A_1^2(\varphi_n - \varphi_m),\ \varphi_n - \varphi_m) \leqslant$$
$$\leqslant \|A^2(\varphi_n - \varphi_m)\|_M \|\varphi_n - \varphi_m\|_N.$$

The first factor in the right member tends to zero since the operator A^2 is by assumption completely continuous and transforms every E_N-weakly convergent sequence of functions into a sequence which converges with respect to the norm in L_M^*. The second factor in the right member is bounded.

Consequently, the sequence $A_1\varphi_n(x)$ $(n = 1, 2, \cdots)$ converges in norm in L^2.

Now let $\psi_1(x), \psi_2(x), \cdots$ be an arbitrary sequence which converges weakly in E_N. Since L^2 is dense in E_N, a sequence $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ can be found in L^2 such that $\|\psi_n - \varphi_n\|_N \to 0$. The sequence $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ is also weakly convergent. As was already

proved, the sequence $\boldsymbol{A}_1\varphi_n(x)$ converges in norm in L^2. Therefore, the sequence $\boldsymbol{A}_1\psi_n(x)$ also converges in norm in L^2 inasmuch as

$$\|\boldsymbol{A}_1(\psi_n - \psi_m)\|_{L^2} \leqslant \|\boldsymbol{A}_1(\varphi_n - \varphi_m)\|_{L^2} + \\ + \|\boldsymbol{A}_1(\varphi_n - \psi_n)\|_{L^2} + \|\boldsymbol{A}_1(\varphi_m - \psi_m)\|_{L^2}. \bigstar$$

Following a line of reasoning analogous to that used in the proof of Theorem 15.6, we arrive at the following proposition.

THEOREM 16.8. *Suppose the positive definite self-adjoint completely continuous linear operator $\boldsymbol{A}$ in L^2 admits of a continuous extension to an operator $\overline{\boldsymbol{A}} \in \{E_N \to L^*_M$; comp. c.$\}$, where $N(u) \prec u^2 \prec M(u)$. Then the operator $\boldsymbol{A}$ splits, i.e. $\overline{\boldsymbol{A}} = \boldsymbol{H}\boldsymbol{H}^*$, where $\boldsymbol{H} \in \{L^2 \to L^*_M$; comp. c.$\}$ and $\boldsymbol{H}^*$ is the operator from $\{E_N \to L^2$; comp. c.$\}$ which is adjoint to $\boldsymbol{H}$.*

7. *On operators of potential type.* We shall assume that G is a closed bounded region (with a sufficiently smooth boundary) in n-dimensional space.

An operator of potential type is a linear integral operator whose kernel is symmetric and satisfies the condition

$$|k(x, y)| \leqslant \frac{a}{r^\lambda} \quad (x, y \in G), \tag{16.13}$$

where r is the distance between the points x and y.

A detailed theory of operators of potential type is developed in S. SOBOLEV [1] in connection with embedding theorems. S. Sobolev and other authors showed that under definite relations among the numbers λ, α and the dimension n of the space, an operator of potential type acts from the space L^α into a definite space L^{α_1} or into the space C of continuous functions. In particular, for $\lambda = n/2$, an operator of potential type acts, according to S. Sobolev's theorems, from the space L^2 into an arbitrary space L^{α_1} $(\alpha_1 > 1)$.

Theorems on the splitting of linear operators enable one to strengthen this assertion.

Let $\boldsymbol{A}$ be a linear integral operator acting in L^2 whose kernel satisfies condition (16.13) in which $\lambda = n/2$. Then the operator $\boldsymbol{A}^2$ will also be a linear integral operator with kernel

$$k_2(x, y) = \int_G k(x, z)k(z, y)\,dz.$$

A direct calculation, using condition (16.13), shows that

$$|k_2(x, y)| \leqslant b + c|\ln|r||. \tag{16.14}$$

Consequently, $k_2(x, y) \in \hat{L}^*_{\Psi}$, where $\Psi(u) = e^{|u|} - |u| - 1$.

In virtue of Theorem 15.4, the operator $\bar{A}^2$, defined by the kernel $k_2(x, y)$, acts from the space $E_{\Phi} = L^*_{\Phi}$, where $\Phi(u) = (1 + |u|)\ln(1 + |u|) - |u|$, into L^*_{Ψ} and is a continuous operator. $\bar{A}^2$ is a continuous extension of the operator A^2. Applying Theorem 15.5, we arrive at the conclusion that $A \in \{L^2 \to L^*_{\Psi};\ \text{c.}\}$.

Let $\Psi_1(u) = |u|(e^{|u|^{1-\varepsilon}} - 1)$, where $0 < \varepsilon < 1$. It is easily seen that $\Psi_1(u)$ satisfies the Δ^2-condition. Therefore, the complementary function $\Phi_1(u)$ satisfies the Δ'-condition. The kernel $k_2(x, y)$, constructed above, belongs to the space $\hat{E}_{\Psi_1}$ since

$$\int_G \int \Psi_1\{|k_2(x, y)|^{1/(1-\varepsilon)}\}\,dx\,dy < \infty.$$

In virtue of Theorem 16.1, the operator $\bar{A}^2$ acts from $E_{\Phi_1} = L^*_{\Phi_1}$ into $L^*_{\Psi_1}$. Applying Theorem 16.7, we arrive at the conclusion that $A \in \{L^2 \to L^*_{\Psi_1};\ \text{comp. c.}\}$.

Thus, the following theorem is valid.

THEOREM 16.9. *Suppose the symmetric kernel $k(x, y)$ satisfies the condition* (16.13), *where $\lambda = n/2$. Then the linear integral operator A, defined by the kernel $k(x, y)$, acts from L^2 into L^*_{Ψ}, where $\Psi(u) = e^{|u|} - |u| - 1$, and is continuous — and, as an operator acting from L^2 into $L^*_{\Psi_1}$, where $\Psi_1(u) = |u|(e^{|u|^{1-\varepsilon}} - 1)$ $(0 < \varepsilon < 1)$, it is completely continuous.*

As is known, the Green function $k(x, y)$ of the Laplace operator (in the case of homogeneous boundary conditions) for a four-dimensional region satisfies the inequality $|k(x, y)| \leqslant a/r^2$.

We find ourselves in a situation where the conditions of Theorem 16.9 are satisfied. Consequently, the Green function in the case of a four-dimensional region generates a linear integral operator belonging to $\{L^2 \to L^*_{\Psi};\ \text{c.}\}$ and $\{L^2 \to L^*_{\Psi_1};\ \text{comp. c.}\}$, where $\Psi(u)$ and $\Psi_1(u)$ are the N-functions defined above.

We note one more fact. The Green function of the Laplace operator for a two-dimensional region satisfies the condition $|k(x, y)| < b + c|\ln|r||$. It follows from Theorem 15.6 that the linear operator A, defined by this Green's function, is splittable

in the sense that it is representable in the form $A = HH^*$, where $H \in \{L^2 \to L^*_\Psi;\ \text{c.}\}$ and H^* is the operator in $\{E_\Phi \to L^2;\ \text{c.}\}$ which is adjoint to H.

An analogous representation $A = HH^*$, where $H \in \{L^2 \to L^*_{\Psi_1};\ \text{comp. c.}\}$, $H^* \in \{E_{\Phi_1} \to L^2;\ \text{comp. c.}\}$, follows from Theorem 16.8.

In the last two assertions, the N-functions $\Psi(u)$ and $\Psi_1(u)$ are the functions indicated in the formula of Theorem 16.9.

§ 17. Simplest nonlinear operator

1. *The Carathéodory condition.* We say that the real-valued function $f(x, u)$ of the two variables $x \in G$, $-\infty < u < \infty$, satisfies the *Carathéodory condition* if it is continuous in u for almost all $x \in G$ and measurable in x for every fixed u.

The following theorem holds.

THEOREM 17.1. *A function $f(x, u)$ satisfies the Carathéodory condition if, and only if, to a prescribed $\varepsilon > 0$ there corresponds a closed set $G_0 \subset G$ such that mes $(G \setminus G_0) < \varepsilon$ and the function $f(x, u)$ is continuous in all variables on the set $G_0 \times (-\infty, \infty)$.*

The sufficiency of the condition in the theorem is obvious. For a proof of the necessity, see KRASNOSEL'SKIĬ and LADYŽENSKIĬ [1] and KRASNOSEL'SKIĬ [5].

We shall denote by f the operator defined on the set of real-valued functions $u(x)$ $(x \in G)$ by means of the equality

$$fu(x) = f[x, u(x)]. \tag{17.1}$$

If the Carathéodory condition is satisfied, then, in virtue of Theorem 17.1, the operator f transforms measurable functions into measurable functions — and it transforms sequences of functions which are convergent in measure into sequences of functions which are also convergent in measure.

In the sequel, when we consider operators defined by formula (17.1), we shall always assume that the corresponding functions satisfy the Carathéodory condition.

2. *Domain of definition of the operator f.* We recall that $\Pi(E_M, r)$ denotes the totality of functions $u(x) \in L^*_M$ for which $d(u, E_M) < r$ (see p. 82). We shall denote the sphere of radius r and center at the point u of the Banach space E by $T(u, r; E)$.

LEMMA 17.1. *Let*

$$f_1(x, 0) \equiv 0 \quad (x \in G). \tag{17.2}$$

Suppose the operator $\boldsymbol{f}_1$ acts from a sphere $T(\theta, r; L^*_{M_1})$ into the space $L^*_{M_2}$, the class L_{M_2} or E_{M_2}. Then the operator $\boldsymbol{f}_1$ acts from $\Pi(E_{M_1}, r)$ into the space $L^*_{M_2}$, the class L_{M_2} or E_{M_2}, respectively. If the operator $\boldsymbol{f}_1$ acts from the sphere $T(\theta, r; E_{M_1})$ into $L^*_{M_2}$, L_{M_2} or E_{M_2}, then it acts from all of E_{M_1} into $L^*_{M_2}$, L_{M_2} or E_{M_2}, respectively.

PROOF. Let $u(x) \in \Pi(E_{M_1}, r)$. In virtue of Lemma 10.1, a set $G_0 \subset G$ can be found such that $u(x)\kappa(x; G_0) \in E_{M_1}$ and $\|u_0\|_{M_1} < r$, where $u_0(x) = u(x)\kappa(x; G \setminus G_0)$. The function $u(x)\kappa(x; G_0)$ has an absolutely continuous norm. Therefore, the function $u(x)$ can be written in the form

$$u(x) = u_0(x) + u_1(x) + \cdots + u_k(x),$$

where $\|u_i\|_{M_1} < r$ $(i = 0, 1, \cdots, k)$, and, for $i \neq j$,

$$u_i(x)u_j(x) = 0 \quad (x \in G).$$

In virtue of (17.2),

$$\boldsymbol{f}_1 u_i(x) \boldsymbol{f}_1 u_j(x) = 0 \quad (x \in G,\ i \neq j), \tag{17.3}$$

and

$$\boldsymbol{f}_1 u(x) = \boldsymbol{f}_1 u_0(x) + \boldsymbol{f}_1 u_1(x) + \cdots + \boldsymbol{f}_1 u_k(x). \tag{17.4}$$

If $\boldsymbol{f}_1 T(\theta, r; L^*_{M_1}) \subset L^*_{M_2}$, then each of the terms in the right member of formula (17.4) is a function in $L^*_{M_2}$. Therefore, $\boldsymbol{f}_1 u(x)$ also belongs to $L^*_{M_2}$.

If $\boldsymbol{f}_1 T(\theta, r; L^*_{M_1}) \subset L_{M_2}$, then, in virtue of (17.3),

$$\int_G M[\boldsymbol{f}_1 u(x)]\,dx = \sum_{i=0}^{k} \int_G M[\boldsymbol{f}_1 u_i(x)]\,dx < \infty,$$

i.e. $\boldsymbol{f}_1 u(x)$ also belongs to L_{M_2}.

If $\boldsymbol{f}_1 T(\theta, r; L^*_{M_1}) \subset E_{M_2}$, then all the terms in the right member of (17.4) are functions in E_{M_2}. Therefore, $\boldsymbol{f}_1 u(x)$ is also an element in E_{M_2}.

Finally, if $\boldsymbol{f}_1$ is considered only on the functions in E_{M_1}, then $u_0(x) \equiv 0$. Therefore, $\boldsymbol{f}_1 T(\theta, r; E_{M_1}) \subset L^*_{M_2}$ implies that all the

terms in the right member of (17.4) belong to $L^*_{M_2}$ and, consequently, $\boldsymbol{f}_1 u(x)$ is also an element in $L^*_{M_2}$.

The remaining cases are considered analogously. ★

Let us now assume that the N-function $M_1(u)$ satisfies the Δ_2-condition. It then follows from Lemma 17.1 that the operator $\boldsymbol{f}_1$ is defined on the entire space $L^*_{M_1} = L_{M_1} = E_{M_1}$ provided it is defined on any sphere of this space. The question naturally arises whether or not this fact holds in the case when $M_1(u)$ does not satisfy the Δ_2-condition. It turns out that this is not so.

In fact, if $M_1(u)$ does not satisfy the Δ_2-condition, then the operator

$$\boldsymbol{f}_1 u(x) = M_2^{-1}\{M_1[u(x)]\} \tag{17.5}$$

acts from the unit sphere in the space $L^*_{M_1}$ into L_{M_2} since

$$\int_G M_2[\boldsymbol{f}_1 u(x)]\,dx = \int_G M_1[u(x)]\,dx \leqslant \|u\|_{M_1} \leqslant 1$$

for $\|u\|_{M_1} \leqslant 1$. However, in an arbitrary sphere of radius $r \leqslant 1$, a function $u(x)$ can be found such that $\boldsymbol{f}_1 u(x) \overline{\in} L_{M_2}$. To this end, it suffices to choose a function $u(x)$, with norm $\|u\|_{M_1} < r$, which does not belong to L_{M_1}.

If the N-function $M_2(u)$ satisfies the Δ_2-condition, then the operator (17.5) from the unit sphere of the space $L^*_{M_1}$ acts simultaneously into $L^*_{M_2}$, L_{M_2} and E_{M_2} since $L^*_{M_2} = L_{M_2} = E_{M_2}$. However, not all its values on a sphere of larger radius belong to $L^*_{M_2}$.

THEOREM 17.2. *Suppose the operator $\boldsymbol{f}$ acts from a sphere $T(\theta, r; L^*_{M_1})$ into the space L_{M_2} or into the space E_{M_2}. Then the operator $\boldsymbol{f}$ acts from $\Pi(E_{M_1}, r)$ into the space $L^*_{M_2}$ or into the space E_{M_2}, respectively. If the operator $\boldsymbol{f}$ acts from the sphere $T(\theta, r; E_M)$ into $L^*_{M_2}$ or into E_{M_2}, then it acts from all of E_{M_1} into $L^*_{M_2}$ or into E_{M_2}, respectively.*

PROOF. The function $f_1(x, u) = f(x, u) - f(x, 0)$ satisfies condition (17.2). Therefore, Lemma 17.1 can be applied to the operator $\boldsymbol{f}_1 u(x) = f_1[x, u(x)]$. This implies the assertion of the theorem. ★

3. *Theorems on continuity.* A nonlinear operator $\boldsymbol{B}$, acting from the Banach space E_1 into the Banach space E_2, is said to be continuous at the point $u_0 \in E_1$ if

$$\lim_{\substack{u \in E_1 \\ \|u - u_0\|_{E_1} \to 0}} \|\boldsymbol{B}u - \boldsymbol{B}u_0\|_{E_2} = 0.$$

For nonlinear operators, in distinction from the case of linear operators, continuity at one point does not imply continuity at other points. In general, a nonlinear operator can be defined, as we saw in the example of the operator $\boldsymbol{f}$, only on a part of the space E_1.

A nonlinear operator $\boldsymbol{B}$ is said to be bounded on the sphere T of the space E_1 if

$$\sup_{x \in T} \|\boldsymbol{B}u\|_{E_2} < \infty.$$

In distinction from linear operators, the concepts of boundedness and continuity are not connected with one another in the case of nonlinear operators: there exist discontinuous operators which are bounded and, conversely, there exist operators which are continuous on the entire space but which are not bounded on any spheres.

In the sequel, we shall find the following assertion from KRASNOSEL'SKIĬ [5], pp. 31, 35, 36, useful.

LEMMA 17.2. *Suppose the operator* $\boldsymbol{g}$ *acts from the space* L *of summable functions into the space* L. *Then the operator* $\boldsymbol{g}$ *is continuous and bounded with respect to the norm of the space* L *and the function* $g(x, u)$ *satisfies the condition*

$$|g(x, u)| \leqslant a(x) + b|u| \quad (x \in G, -\infty < u < \infty),$$

where $a(x) \in L$ *and* b *is a positive number.*

THEOREM 17.3. *If the operator* $\boldsymbol{f}$ *acts from* $\Pi(E_{M_1}, r)$ *into* E_{M_2}, *then* $\boldsymbol{f}$ *is continuous at every point of* $\Pi(E_{M_1}, r)$.

PROOF. We shall first assume that $\boldsymbol{f}\theta = \theta$ and show that the operator $\boldsymbol{f}$ is continuous at the zero θ of the space $L^*_{M_1}$. If we assume the contrary, there exists a sequence of functions $u_n(x) \in L^*_{M_1}$ $(n = 1, 2, \cdots)$ such that

$$\lim_{n \to \infty} \|u_n\|_{M_1} = 0 \tag{17.6}$$

whereas

$$\|\boldsymbol{f}u_n\|_{M_2} > \alpha \quad (n = 1, 2, \cdots), \tag{17.7}$$

where α is some number. It follows from Lemma 9.2 and (17.6) that

$$\lim_{n \to \infty} \int_G M_1\left[\frac{2}{r} u_n(x)\right] dx = 0. \tag{17.8}$$

It follows from (17.7) that

$$\int_G M_2\left[\frac{2}{\alpha}\boldsymbol{f}u_n(x)\right]dx \geqslant \left\|\frac{2}{\alpha}\boldsymbol{f}u_n\right\|_{M_2} - 1 > 1. \qquad (17.9)$$

Thus, if the operator $\boldsymbol{f}$ is discontinuous at zero, then there exists a sequence of functions $u_n(x) \in L^*_{M_1}$ such that relations (17.8) and (17.9) hold.

We now consider the operator $\boldsymbol{g}$ defined by means of the formula

$$\boldsymbol{g}u(x) = M_2\left\{\frac{2}{\alpha}\boldsymbol{f}\left(\frac{r}{2}M_1^{-1}[u(x)]\right)\right\}.$$

If $u(x)$ is a summable function, then $M_1^{-1}[u(x)] \in L_{M_1}$. In virtue of Theorem 10.1, $M_1^{-1}[u(x)] \in \Pi(E_{M_1}, 3/2)$. Therefore,

$$\frac{r}{2}M_1^{-1}[u(x)] \in \Pi(E_{M_1}, \tfrac{3}{4}r).$$

By the condition of the theorem, we have that

$$\boldsymbol{f}\left(\frac{r}{2}M_1^{-1}[u(x)]\right) \in E_{M_2}.$$

This means that $\boldsymbol{g}u(x) \in L$. In virtue of Lemma 17.2, the operator $\boldsymbol{g}$, acting from L into L, is continuous at zero. It therefore must follow from (17.8) that

$$\lim_{n\to\infty}\int_G \boldsymbol{g}\left\{M_1\left[\frac{2}{r}u_n(x)\right]\right\}dx = \lim_{n\to\infty}\int_G M_2\left[\frac{2}{\alpha}\boldsymbol{f}u_n(x)\right]dx = 0.$$

We have thus arrived at a contradiction with (17.9).

This means that the operator $\boldsymbol{f}$ is continuous at the zero of the space $L^*_{M_1}$ if $\boldsymbol{f}\theta = \theta$.

We now proceed to the consideration of the general case: we shall show, without any additional assumptions, that the operator $\boldsymbol{f}$ is continuous at an arbitrary point $u_0(x)$ of the set $\Pi(E_{M_1}, r)$. Let $d = d(u_0, E_{M_1})$. Clearly, $d < r$. The continuity of the operator $\boldsymbol{f}$ at the point $u_0(x)$ is equivalent to the continuity of the operator $\boldsymbol{f}_1 u(x) = \boldsymbol{f}[u_0(x) + u(x)] - \boldsymbol{f}u_0(x)$ at the zero of the space $L^*_{M_1}$. The operator $\boldsymbol{f}_1$ acts from the sphere $T(\theta, r - d; L^*_{M_1})$ into E_{M_2}. In virtue of Theorem 17.2, it acts from $\Pi(E_{M_1}, r - d)$ into E_{M_2}.

Since $\boldsymbol{f}_1\theta = \theta$, we have, by what has already been proved, that the operator $\boldsymbol{f}_1$ is continuous at the zero of $L^*_{M_1}$. ★

We note that Theorem 17.3 yields a rough criterion for the continuity of the operator $\boldsymbol{f}$. Even the continuity, for example, of the operator of the identity transformation (i.e. of the operator $\boldsymbol{f}u(x) = u(x)$) in the spaces L^*_M does not follow from it if $M(u)$ does not satisfy the Δ_2-condition.

If, under the conditions of Theorem 17.3, the N-function $M_2(u)$ satisfies the Δ_2-condition, then this theorem signifies that the operator $\boldsymbol{f}$ is continuous if it acts into the space $L^*_{M_2} = L_{M_2} = E_{M_2}$. The question arises whether the operator $\boldsymbol{f}$ is always continuous if it acts from $\Pi(E_{M_1}, r)$ into the space $L^*_{M_2}$ when $M_2(u)$ does not satisfy the Δ_2-condition. It turns out that the operator $\boldsymbol{f}$ is not always continuous. Here is an example. Let

$$\boldsymbol{f}u(x) = M_2^{-1}\{M_1[u(x)]\}, \tag{17.10}$$

where $M_1(u)$ satisfies but $M_2(u)$ does not satisfy the Δ_2-condition. Clearly, the operator $\boldsymbol{f}$ acts from $\Pi(E_{M_1}, 1) = L_{M_1}$ into L_{M_2}. Suppose $v(x)$ belongs to L_{M_2} but that $v(x) \bar{\in} E_{M_2}$. Then, in virtue of Lemma 10.1, we have that

$$\lim_{n\to\infty} \|v - v_n\|_{M_2} = d(v, E_{M_2}) = d > 0, \tag{17.11}$$

where

$$v_n(x) = \begin{cases} v(x) & \text{if } |v(x)| \leqslant n, \\ 0 & \text{if } |v(x)| > n. \end{cases} \tag{17.12}$$

The functions $u_n(x) = M_1^{-1}\{M_2[v(x) - v_n(x)]\}$ belong to L_{M_1}, with

$$\lim_{n\to\infty} \int_G M_1[u_n(x)]\,dx = \lim_{n\to\infty} \int_G M_2[v(x) - v_n(x)]\,dx = 0.$$

Since, in L_{M_1}, mean convergence implies convergence in norm, we have that $\lim_{n\to\infty} \|u_n\|_{M_1} = 0$. From this equality and (17.11), it follows that the operator $\boldsymbol{f}$ is discontinuous at the zero of the space $L^*_{M_1}$ since $\boldsymbol{f}u_n(x) = v(x) - v_n(x)$.

4. *Boundedness of the operator* $\boldsymbol{f}$. The boundedness of the set of values of the operator $\boldsymbol{f}$ on a sphere of the space $L^*_{M_1}$ can be

proved under fewer restrictions that those under which the continuity of the operator $\boldsymbol{f}$ was proved in Theorem 17.3.

THEOREM 17.4. *Suppose the operator* $\boldsymbol{f}$ *acts from the sphere* $T(\theta, r; L^*_{M_1})$ *into the class* L_{M_2}. *Then* $\boldsymbol{f}$ *is bounded on any sphere* $T(\theta, r_1; L^*_{M_1})$ $(r_1 < r)$: $\sup_{\|u\|_{M_1} < r_1} \|\boldsymbol{f}u\|_{M_2} < \infty$.

PROOF. In virtue of Lemma 17.1, the operator $[\boldsymbol{f}u(x) - \boldsymbol{f}\theta]/2$ acts from $\Pi(E_{M_1}, r)$ into the class L_{M_2}. Therefore the operator

$$\boldsymbol{g}v(x) = M_2\{\tfrac{1}{2}\boldsymbol{f}(r_1 M_1^{-1}[v(x)]) - \tfrac{1}{2}\boldsymbol{f}\theta\}$$

acts from L into L.

For functions $u(x) \in T(\theta, r_1; L^*_{M_1})$ we have that

$$\int_G M_1\left[\frac{u(x)}{r_1}\right] dx \leqslant 1.$$

In virtue of Lemma 17.2,

$$\sup_{\|u\|_{M_1} < r_1} \int_G \boldsymbol{g}\left\{M_1\left[\frac{u(x)}{r_1}\right]\right\} dx < \infty,$$

i.e.

$$\sup_{\|u\|_{M_1} < r_1} \int_G M_2[\tfrac{1}{2}\boldsymbol{f}u(x) - \tfrac{1}{2}\boldsymbol{f}\theta]\, dx < \infty.$$

Consequently,

$$\sup_{\|u\|_{M_1} < r_1} \|\boldsymbol{f}u\|_{M_2} \leqslant \|\boldsymbol{f}\theta\|_{M_2} + \sup_{\|u\|_{M_1} < r_1} \|\boldsymbol{f}u - \boldsymbol{f}\theta\|_{M_2} \leqslant$$

$$\leqslant \|\boldsymbol{f}\theta\|_{M_2} + 2\left(1 + \sup_{\|u\|_{M_1} < r_1} \int_G M_2[\tfrac{1}{2}\boldsymbol{f}u(x) - \tfrac{1}{2}\boldsymbol{f}\theta]\, dx\right) < \infty. \bigstar$$

The question arises whether the operator $\boldsymbol{f}$ is bounded on spheres with larger radii. It turns out that this is not necessarily so. As an example, we consider again the operator (17.10) in which connection we shall assume that $M_1(u)$ does not satisfy but $M_2(u)$ satisfies the Δ_2-condition.

The operator (17.10) acts from the class L_{M_1} into $L_{M_2} = E_{M_2}$ and, in virtue of Theorem 17.3, it is continuous on E_{M_1}. We shall show that the values of this operator $\boldsymbol{f}$ are not bounded on every sphere $T(\theta, 1 + \varepsilon; E_{M_1})$, where ε is an arbitrary positive number.

Since $M_2(u)$ satisfies the Δ_2-condition, boundedness in norm is equivalent to boundedness in the mean. It therefore suffices to show that there exists a sequence of functions $u_n(x) \in T(\theta, 1+\varepsilon; E_{M_1})$ such that

$$\sup_{\|u_n\|_{M_1}<1+\varepsilon} \int_G M_2[\boldsymbol{f}u_n(x)]\,dx = \sup_{\|u_n\|_{M_1}<1+\varepsilon} \int_G M_1[u_n(x)]\,dx = \infty. \qquad (17.13)$$

The functions $u_n(x)$ are defined by equalities similar to (17.12) where $u(x)$ is a function not belonging to L_{M_1} such that $\|u\|_{M_1} < 1 + \varepsilon$. The existence of the function $u(x)$ follows from Theorem 10.1. If relation (17.13) were not satisfied for the functions $u_n(x)$, then, in virtue of Fatou's theorem (see p. 71), the function $u(x)$ would belong to L_{M_1}.

5. *General form of the operator* $\boldsymbol{f}$.

THEOREM 17.5. *The operator* $\boldsymbol{f}$ *acts from the class* L_{M_1} *into the class* L_{M_2} *if, and only if,*

$$M_2[f(x, u)] \leqslant a(x) + bM_1(u) \quad (x \in G, -\infty < u < \infty), \qquad (17.14)$$

where $a(x) \in L$, $b \geqslant 0$.

PROOF. If $\boldsymbol{f}$ acts from L_{M_1} into L_{M_2}, then the operator $\boldsymbol{g}v(x) = M_2\{\boldsymbol{f}(M_1^{-1}[v(x)])\}$ acts from L into L and, in virtue of Lemma 17.2, we have that. $M_2\{f[x, M_1^{-1}(v)]\} \leqslant a(x) + b|v|$. Setting $v = M(u)$ in the last inequality, we obtain (17.14). The sufficiency of condition (17.14) is obvious. ★

We now assume that condition (17.14) is satisfied. Then, in virtue of (1.20), we have that

$$|f(x, u)| \leqslant c(x) + b_1 M_2^{-1}[M_1(u)] \quad (x \in G, -\infty < u < \infty), \qquad (17.15)$$

where $c(x) = M_2^{-1}[a(x)] \in L_{M_2}$. Thus, condition (17.15) is necessary in order that the operator $\boldsymbol{f}$ act from L_{M_1} into L_{M_2}. In the case when the N-function $M_2(u)$ satisfies the Δ_2-condition, condition (17.15) is at the same time a sufficient condition inasmuch as in this case (17.15) implies (17.14).

6. *Sufficient conditions for the continuity and boundedness of the operator* $\boldsymbol{f}$. The assertions proved above imply the next theorem.

THEOREM 17.6. *Suppose the function $f(x, u)$ satisfies the inequality*

$$|f(x, u)| \leqslant c(x) + b_1 M_2^{-1}\left[M_1\left(\frac{u}{r}\right)\right] \qquad (17.16)$$

$$(x \in G, \ -\infty < u < \infty),$$

where $c(x) \in L_{M_2}$, $b_1 \geqslant 0$, in which connection the N-function $M_2(u)$ satisfies the Δ_2-condition. Then the operator $\boldsymbol{f}$ acts from $\Pi(E_{M_1}, r)$ into the space $L^*_{M_2} = E_{M_2}$, is continuous at all points of $\Pi(E_{M_1}, r)$, and is bounded on every sphere $T(\theta, r_1; L^*_{M_1})$ $(r_1 < r)$.

If the N-function $M_2(u)$ does not satisfy the Δ_2-condition, then one must require, in order to have continuity of the operator $\boldsymbol{f}$, that the set of its values belong to E_{M_2}. Therefore, we must place more severe restrictions on the function $f(x, u)$.

THEOREM 17.7. *Suppose the function $f(x, u)$ satisfies the inequality*

$$|f(x, u)| \leqslant b(x) + aQ^{-1}\left\{M_2^{-1}\left[M_1\left(\frac{u}{r}\right)\right]\right\} \qquad (17.17)$$

$$(x \in G, \ -\infty < u < \infty),$$

where $b(x) \in E_{M_2}$, $Q(u)$ is an N-function, and a, r are positive numbers. Then the operator $\boldsymbol{f}$ acts from $\Pi(E_{M_1}, r)$ into E_{M_2}, is continuous at all points of $\Pi(E_{M_1}, r)$, and is bounded on every sphere $T(\theta, r_1; L^*_{M_1})$ $(r_1 < r)$.

7. *The operator $\boldsymbol{f}$ and E_N-weak convergence.* We proved the continuity of the operator $\boldsymbol{f}$ under the assumption that it acts from the sphere $T(\theta, r; L^*_{M_1})$ into E_{M_2}. The boundedness of the operator $\boldsymbol{f}$ was established under weaker hypotheses. We shall show that the continuity of the operator $\boldsymbol{f}$ in some weakened sense can be proved under these weaker assumptions.

THEOREM 17.8. *Suppose the operator $\boldsymbol{f}$ acts from the sphere $T(\theta, r; L^*_{M_1})$ into the class L_{M_2}. Then this operator transforms an arbitrary sequence $u_n(x) \in T(\theta, r; L^*_{M_1})$, which converges in norm in the space $L^*_{M_1}$ to the interior point $u_0(x)$ of this sphere, into the sequence of functions $\boldsymbol{f}u_n(x)$ which is E_{N_2}-weakly convergent in the space $L^*_{M_2}$ to the function $\boldsymbol{f}u_0(x)$.*

PROOF. Since the functions $u_n(x)$ converge in norm to the function $u_0(x)$, they also converge to this function in measure.

Therefore, the sequence of functions $\boldsymbol{f}u_n(x)$ also converges in norm to the function $\boldsymbol{f}u_0(x)$. Since the function $u_0(x)$ is an interior point of the sphere $T(\theta, r; L^*_{M_1})$, we may assume, without loss of generality, that all $u_n(x) \in T(\theta, r_1; L^*_{M_1})$, where $r_1 < r$. It follows from Theorem 17.4 that the norms of the functions $\boldsymbol{f}u_n(x)$ are uniformly bounded. In virtue of Theorem 14.6, this sequence is E_N-weakly convergent to the function $\boldsymbol{f}u_0(x)$. ★

§ 18. Differentiability. Gradient of the norm

1. *Differentiable functionals.* We shall consider real-valued functionals defined on Orlicz spaces L^*_M (or on parts of these spaces).

We say that the functional $\boldsymbol{F}(u)$ is *differentiable* at the point $u_0(x)$, which is an interior point of the domain of definition of this functional, if the increment of the functional can be written in the form

$$\boldsymbol{F}(u_0 + h) - \boldsymbol{F}(u_0) = \boldsymbol{l}_{u_0}(h) + \omega(u_0, h), \tag{18.1}$$

where $\boldsymbol{l}_{u_0}(h)$ is a linear functional on L^*_M and the remainder $\omega(u_0, h)$ satisfies the condition

$$\lim_{\|h\|_M \to 0} \frac{\omega(u_0, h)}{\|h\|_M} = 0. \tag{18.2}$$

The functional $\boldsymbol{F}(u)$ is differentiable on the set $\mathfrak{M}$ if it is differentiable at every point of $\mathfrak{M}$.

The operator $\boldsymbol{\Gamma}$ which assigns the functional $\boldsymbol{l}_{u_0}$ to the element $u_0(x) \in \mathfrak{M}$ is called the *gradient* of the functional $\boldsymbol{F}(u)$. The gradient acts from the space L^*_M into the conjugate space. The space L^*_N, where $N(v)$ is the complementary function to $M(u)$, can be considered as a part of the conjugate space. In the case when the gradient acts into L^*_N, equality (18.1) can be rewritten in the form

$$\boldsymbol{F}(u_0 + h) - \boldsymbol{F}(u_0) = (\boldsymbol{\Gamma}u_0, h) + \omega(u_0, h), \tag{18.3}$$

where, as usual, the symbol (w, h) denotes the inner product:

$$(w, h) = \int_G w(x)h(x)\,dx \qquad (w(x) \in L^*_N,\ h(x) \in L^*_M).$$

We shall retain the terminology gradient for the operator which assigns the function $\boldsymbol{\Gamma}u_0(x)$ to the function $u_0(x)$.

2. *Measurability of the function* $\theta(x)$. Suppose the function $F(x, u)$ $(x \in G, \ -\infty < u < \infty)$ satisfies the Carathéodory conditions and has a continuous derivative with respect to u: $F'_u(x, u)$. Clearly, the function $F'_u(x, u)$ also satisfies the Carathéodory conditions.

Suppose $u(x)$ is a measurable function on G. On the basis of the mean-value theorem, there exists a function $\theta(x)$ such that

$$F(x, u(x)) - F(x, 0) = u(x)F'_u(x, \theta(x)u(x)). \tag{18.4}$$

Generally speaking, the function $\theta(x)$ is not uniquely defined. We shall use the following assertion in the sequel.

LEMMA 18.1. *There exists a measurable function* $\theta(x)$, *satisfying the inequalities*

$$0 \leqslant \theta(x) \leqslant 1, \tag{18.5}$$

for which equality (17.21) *is satisfied.*

PROOF. We define the function $\theta(x)$ for all $x \in G$ as the minimum of those $\theta \in [0, 1]$ for which the equality

$$F(x, u(x)) - F(x, 0) = u(x)F'_u(x, \theta u(x))$$

is satisfied. This minimum exists inasmuch as $F'_u(x, u)$ is continuous with respect to u.

Let $\varepsilon > 0$ be given. In virtue of Luzin's C-property and Theorem 17.1, a set $G_1 \subset G$ can be found such that mes $(G \setminus G_1) < \varepsilon$ and the functions $F'_u(x, u)$, $F(x, 0)$, $F(x, u)$ and $u(x)$ are continuous in all variables simultaneously for $x \in G_1$, $-\infty < u < \infty$. Since ε is arbitrary, it will suffice to prove the measurability of $\theta(x)$ on G_1.

Let $x_0 \in G_1$, $x_n \in G_1$ $(n = 1, 2, \cdots)$ and

$$\lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} \theta(x_n) = \theta_0.$$

Passing to the limit in the equality

$$F(x_n, u(x_n)) - F(x_n, 0) = u(x_n)F'_u(x_n, \theta(x_n)u(x_n)),$$

we obtain that

$$F(x_0, u(x_0)) - F(x_0, 0) = u(x_0)F'_u(x_0, \theta_0 u(x_0)).$$

The last equality implies that $\theta(x_0) \leqslant \theta_0$.

This means that the function $\theta(x)$ is lower semi-continuous on G_1 and, consequently, that it is measurable. ★

3. *Functional for the operator* $\boldsymbol{f}$. We now consider the functional

$$\boldsymbol{F}_1(u) = \int_G dx \int_0^{u(x)} f(x, s)\, ds. \tag{18.6}$$

We shall be interested in the case when the functional $\boldsymbol{F}_1(u)$ is defined on a part of the space L_M^* which is a part of the space L^2. In this case, the N-function $M(u)$ and the function $N(v)$ complementary to it are connected by the relations $N(u) \prec u^2 \prec M(u)$. Since $N(u)$ does not increase more rapidly than u^2, it will satisfy the Δ_2-condition in the fundamental cases. It is under this assumption that the following theorem is proved.

THEOREM 18.1. *Let the operator* $\boldsymbol{f}$ *act from* $\Pi(E_M, r)$ *into the space* $L_N^* = L_N$. *Then the functional* (18.6) *is defined and differentiable on* $\Pi(E_M, r)$, *in which connection its gradient is the operator* $\boldsymbol{f}$.

PROOF. Let $u(x) \in \Pi(E_{M_1}, r)$. In virtue of Lemma 18.1, a measurable function $\theta(x)$ can be found, satisfying condition (18.5), such that

$$\boldsymbol{F}_1(u) = \int_G dx \int_0^{u(x)} f(x, s)\, ds = \int_G f[x, \theta(x)u(x)]u(x)\, dx.$$

Clearly, $\theta(x)u(x) \in \Pi(E_M, r)$. Therefore, $f[x, \theta(x)u(x)] \in L_N$. Consequently, $|\boldsymbol{F}_1(u)| \leqslant \|\boldsymbol{f}\theta u\|_N \|u\|_M < \infty$, from which it follows that the functional $\boldsymbol{F}_1(u)$ is defined on $\Pi(E_M, r)$.

In virtue of the same Lemma 18.1, to each function $h(x) \in L_M^*$ such that $\|h\|_M < r - d(u, E_M)$ there corresponds a measurable function $\theta_h(x)$, satisfying condition (18.5), such that

$$|\boldsymbol{F}_1(u + h) - \boldsymbol{F}_1(u) - (\boldsymbol{f}u, h)| =$$

$$= \left| \int_G [f(x, u(x) + \theta_h(x)h(x)) - f(x, u(x))]h(x)\, dx \right| \leqslant$$

$$\leqslant \|\boldsymbol{f}(u + \theta_h h) - \boldsymbol{f}u\|_N \|h\|_M.$$

It follows from this inequality and the continuity of $\boldsymbol{f}$ (see Theorem 17.3) that $\boldsymbol{f}$ is the gradient of the functional $\boldsymbol{F}_1(u)$. ★

4. *The linear operator* $\boldsymbol{f}$. The linear operator $\boldsymbol{f}$:

$$\boldsymbol{f}u(x) = a(x)u(x), \tag{18.7}$$

where $a(x)$ is a fixed function, is one of the simplest forms of the operators $\boldsymbol{f}$. This operator was, in essence, already studied in § 13 (subsections 4 and 5).

If the function $a(x)$ is bounded, then, obviously, the operator $\boldsymbol{f}$ transforms every Orlicz space into itself and is a continuous operator.

The next assertion follows from Theorem 13.7, Theorem 13.8 and inequalities (13.19) and (13.27).

THEOREM 18.2. *Let $a(x) \in L^*_{\Phi}$. A sufficient condition for the operator* (18.7) *to act from $L^*_{M_1}$ into $L^*_{M_2}$ and be continuous is that there exist mutually complementary N-functions $R(u)$ and $Q(u)$ such that*

$$R(u) < M_2^{-1}[M_1(\alpha u)], \quad Q(u) < M_2^{-1}[\Phi(\alpha u)] \tag{18.8}$$

or, if the N-function $M_2(u)$ satisfies the Δ'-condition, that

$$R(u) < M_1[M_2^{-1}(\alpha u)], \quad Q(u) < \Phi[M_2^{-1}(\alpha u)] \tag{18.9}$$

for large values of the argument. Under these conditions, we have that $\|au\|_{M_2} \leqslant k\|a\|_{\Phi}\|u\|_{M_1}$, where the constant k does not depend on the function $u(x)$.

Under the conditions of this theorem, α is a positive number.

5. *The Fréchet derivative.* Suppose the nonlinear operator $\boldsymbol{A}$ acts from the Banach space E into the Banach space E_1. We say that the linear operator $\boldsymbol{B}$ is the *Fréchet derivative* of the operator $\boldsymbol{A}$ at the point $u_0 \in E$ if $\boldsymbol{A}(u_0 + h) - \boldsymbol{A}u_0 = \boldsymbol{B}h + \omega(u_0, h)$, where $\lim_{\|h\|_E \to 0} \|\omega(u_0, h)\|_{E_1}/\|h\|_E = 0$.

In this connection, the linear operator $\boldsymbol{B}$ also acts from the space E into the space E_1.

The expression $\boldsymbol{B}h$ is called the *Fréchet differential.*

Operators which have a Fréchet derivative are said to be *differentiable.* If an operator is differentiable on some set then it obviously is continuous on this set.

THEOREM 18.3. *Suppose the function $f(x, u)$ together with its derivative $f'_u(x, u)$, which is continuous with respect to u, satisfy the Carathéodory conditions. Suppose, further, that the operator $\boldsymbol{f}$ acts from some sphere $T(u_0, r; L^*_{M_1})$ into the space $L^*_{M_2}$ and that the operator*

$$\boldsymbol{f}_1 u(x) = f'_u(x, u(x)) \tag{18.10}$$

acts from the sphere $T(u_0, r; L^*_{M_1})$ *into the space* L^*_Φ *and is continuous. Finally, let the functions* $M_1(u)$, $M_2(u)$ *and* $\Phi(u)$ *satisfy one of the conditions* (18.8) *or* (18.9). *Then the operator* $\boldsymbol{f}$ *is Fréchet-differentiable at every interior point of the sphere* $T(u_0, r; L^*_{M_1})$, *in which connection the Fréchet differential* $\boldsymbol{B}h$ *at the point* $u(x) \in T$ *is defined by means of the equality* $\boldsymbol{B}h(x) = \boldsymbol{f}_1 u(x)h(x)$, $(h(x) \in L^*_{M_1})$.

PROOF. In virtue of the mean-value theorem, we have that

$$f[x, u(x) + h(x)] - f[x, u(x)] - f'_u[x, u(x)]h(x) = \\ = \{f'_u[x, u(x) + \theta_h(x)h(x)] - f'_u[x, u(x)]\}h(x), \qquad (18.11)$$

where $0 \leqslant \theta_h(x) \leqslant 1$, in which connection the function $\theta_h(x)$ can, in virtue of Lemma 18.1, be considered measurable.

Let $h(x) \in L^*_{M_1}$ and suppose $\|h\|_{M_1}$ is sufficiently small. Then the functions $u(x) + h(x)$ and $u(x) + \theta_h(x)h(x)$ belong to the sphere $T(u_0, r; L^*_{M_1})$. In virtue of Theorem 18.2, $f'_u[x, u(x)]h(x)$ and the entire right member of equality (18.11) belong to the space $L^*_{M_2}$. It follows from this same theorem that

$$\|\boldsymbol{f}(u + h) - \boldsymbol{f}u - \boldsymbol{f}_1 u \cdot h\|_{M_2} \leqslant k\|\boldsymbol{f}_1(u + \theta_h h) - \boldsymbol{f}_1 u\|_\Phi \|h\|_{M_1}.$$

From this, in virtue of the continuity of the operator $\boldsymbol{f}_1$, it follows that

$$\lim_{\|h\|_{M_1} \to 0} \frac{\|\boldsymbol{f}(u + h) - \boldsymbol{f}u - \boldsymbol{f}_1 u \cdot h\|_{M_2}}{\|h\|_{M_1}} = 0. \bigstar$$

As an example, we consider the operator $\boldsymbol{f}u(x) = e^{u(x)}$.

It follows from Theorem 17.6 that the operator $\boldsymbol{f}$ acts from the sphere $T(\theta, \frac{1}{2}; L^*_{M_1})$, where $M_1(u) = e^{|u|} - |u| - 1$, into $L^*_{M_2} = L^2$. In virtue of Theorems 17.2 and 17.3, it acts from $\Pi(E_{M_2}, \frac{1}{2})$ into L^2 and is continuous.

We shall show that this operator is differentiable and that its Fréchet differential $\boldsymbol{B}h(x) = \boldsymbol{f}_1 u(x)h(x)$ at the point $u(x) \in \Pi(E_{M_1}, \frac{1}{2})$ has the form $\boldsymbol{B}h(x) = e^{u(x)}h(x)$.

We apply Theorem 18.3, taking into consideration that $\Phi(u) = |u|^{2+\beta}$, where $0 < \beta < \{1/d(u, E_{M_1})\} - 2$. As T, we consider the sphere with center at the point $u(x)$ and radius

$$r = \{1/(2 + \beta)\} - d(u, E_{M_1}).$$

The operator $\boldsymbol{f}$ acts from the sphere T into L^2 since

$T \subset \Pi(E_{M_1}, \frac{1}{2})$. The operator $\boldsymbol{f}_1 u(x) = e^{u(x)}$ acts from the sphere $T(\theta, 1/(2+\beta); L^*_{M_1})$ into $L^*_\Phi = L^{2+\beta}$. It follows from Theorems 17.2 and 17.3 that it acts from $\Pi(E_{M_1}, 1/(2+\beta))$ and, in particular, from the sphere T into $L^*_\Phi = L^{2+\beta}$ and is continuous.

In order to complete the proof, it suffices to verify that condition (18.8) is satisfied if we set $R(u) = M_1(u)$, $Q(u) = N_1(u)$.

As a second example, we consider the operators $\boldsymbol{f}u(x) = \sin u(x)$, $\boldsymbol{f}_1 u(x) = \cos u(x)$.

These operators act from an arbitrary Orlicz space $L^*_{M_1}$ into every set E_{M_2} and, consequently, they belong to $\{L^*_{M_1} \to L^*_{M_2}; \text{c.}\}$. Let $L^*_{M_1} \subset L^*_{M_2}$. Then the operator $\boldsymbol{B}h(x) = \boldsymbol{f}_1 u(x) \cdot h(x)$ is the Fréchet differential of the operator $\boldsymbol{f}$.

In the examples introduced above, the operator $\boldsymbol{f}$ was differentiable at every point of some sphere T. This explains the simplicity of the proof of differentiability. It is quite easy to give examples of operators which are differentiable at one point of an Orlicz space but which are not differentiable on a set for which the given point is a limit point.

For example, let

$$\boldsymbol{f}u(x) = \sin (e^{u^2(x)} - 1). \tag{18.12}$$

We shall consider $\boldsymbol{f}$ to be an operator in $\{L^4 \to L^2\}$. At those points in which this operator is differentiable, its Fréchet differential obviously has the form

$$\boldsymbol{B}h(x) = 2u(x)e^{u^2(x)} \cos (e^{u^2(x)} - 1)h(x).$$

It is clear that the right member belongs to the space L^2 only for certain functions $u(x)$: for $h(x) \equiv 1$ the right member does not belong to L^2 for a set of functions $u(x)$ which is everywhere dense in L^4.

It will be shown below that the operator (18.12) is differentiable at the zero of the space L^4.

6. *Special condition for differentiability.* In this subsection, a special condition for the differentiability of the operator $\boldsymbol{f}$ at one point of an Orlicz space will be given. We shall restrict ourselves to a condition for the differentiability at the zero θ of the space since the differentiability of the operator $\boldsymbol{g}u(x) = g(x, u(x))$ at the point $u_0(x)$ is equivalent to differentiability at zero of the operator $\boldsymbol{f}u(x) = g[x, u_0(x) + u(x)]$.

Below, we shall consider functions $f(x, u)$ which satisfy condition A):

A) The inequality

$$|f(x, u) - f(x, 0) - f'_u(x, 0)u| \leqslant R(|u|) \tag{18.13}$$

$$(x \in G, \ -\infty < u < \infty),$$

where $R(u)$ is a continuous non-decreasing function such that $R(0) = R'(0) = 0$, holds, in which connection there exists an N-function $P(u)$ which satisfies the Δ'-condition

$$P(uv) \leqslant CP(u)P(v) \tag{18.14}$$

for all values of the argument, such that for sufficiently large values $u_1 \geqslant u_0$, $u_2 \geqslant u_0$ we have that $u_1 < u_2$ implies

$$\frac{R(u_1)}{P(u_1)} < \mu \frac{R(\nu u_2)}{P(\nu u_2)}, \tag{18.15}$$

where μ, ν are positive constants.

As the functions $P(u)$, it is most convenient to consider the functions $P(u) = |u|^r$ $(r > 1)$. Condition (18.15) is satisfied for such functions $P(u)$ if $R(u)$ is the principal part of an N-function satisfying the Δ_2-condition.

In fact, in this case, for large values of the argument, we have that $u^r R(u) < R(\nu u)$, where ν is some constant. Therefore, for large values of u_1 and u_2, it follows from $u_1 < u_2$ that

$$\frac{R(u_1)}{u_1^r} < R(u_1) < R(u_2) = \frac{u_2^r R(u_2)}{u_2^r} < \nu^r \frac{R(\nu u_2)}{(\nu u_2)^r}.$$

In the theorem to be proved below, the fundamental condition of the differentiability of the operator $\boldsymbol{f}$, considered as an operator acting from $L^*_{M_1}$ into $L^*_{M_2}$, will be based on inequality (18.13). A necessary condition for this inequality to guarantee the differentiability of the operator $\boldsymbol{f}$ is, obviously, that

$$\lim_{\|h\|_{M_1} \to 0} \frac{\|R(|h(x)|)\|_{M_2}}{\|h\|_{M_1}} = 0.$$

Considering the characteristic functions of sets whose measures

tend to zero as the functions $h(x)$, we arrive at the condition

$$\lim_{v\to\infty} \frac{N_2^{-1}(v)}{N_1^{-1}(v)} = 0, \tag{18.16}$$

where $N_1(v)$ and $N_2(v)$ are the complementary functions to $M_1(u)$ and $M_2(u)$, since for the characteristic function $\kappa(x)$ of the set G_1 (mes $G_1 = 1/v$), we have that

$$\frac{\|R(\kappa)\|_{M_2}}{\|\kappa\|_{M_1}} = R(1)\,\frac{N_2^{-1}(v)}{N_1^{-1}(v)}.$$

Let $\varepsilon > 0$ be given. It follows from (18.16) that $N_2^{-1}(v) < \varepsilon N_1^{-1}(v)$ for large values of the argument.

From this inequality, it follows, in turn, that the N-function $N_2(v)$ increases essentially more rapidly than the N-function $N_1(v)$. In virtue of Lemma 13.1, the N-function $M_1(u)$ increases essentially more rapidly than $M_2(u)$, i.e. for arbitrary $\varepsilon > 0$, the inequality $M_2(u) < M_1(\varepsilon u)$ is satisfied for large values of the argument.

A sufficient condition that this inequality be satisfied is that the functions $M_1(u)$ and $M_2(u)$ be connected by the relation

$$M_2[Q(u)] \leqslant M_1(u) \quad (u \geqslant u_0), \tag{18.17}$$

where $Q(u)$ is an N-function.

In the sequel, we shall assume that condition (18.17) is satisfied. In this connection, in virtue of Theorem 13.3, a constant $q > 0$ can be found such that

$$\|u\|_{M_2} \leqslant q\|u\|_{M_1} \quad (u(x) \in L^*_{M_1}). \tag{18.18}$$

We consider, further, the condition B):

B) The operator $f_1 h(x) = f'_u(x, 0)h(x)$ acts from $L^*_{M_1}$ into $L^*_{M_2}$ and is continuous.

We note that, in virtue of (18.17), condition B) is always satisfied if the function $a(x) = f'_u(x, 0)$ is bounded. If the function $a(x)$ is not bounded, then a sufficient condition for condition B) to be satisfied is that the function $a(x)$ belong to the space L^*_Φ, where $\Phi(u)$ satisfies one of the conditions in Theorem 18.2.

THEOREM 18.4. *Suppose conditions* A), (18.17) *and* B) *are satisfied. Let*

$$R(u) \leqslant b + aM_2^{-1}[M_1(ku)] \quad (0 \leqslant u < \infty), \tag{18.19}$$

where a, b, $k > 0$. Let $f(x, 0) \in L^*_{M_2}$. Then the operator $\boldsymbol{f}$ acts from some neighborhood of the zero θ of the space $L^*_{M_1}$ into the space $L^*_{M_2}$ and has at the point θ the Fréchet differential

$$\boldsymbol{B}h(x) = \boldsymbol{f}_1 h(x) = f'_u(x, 0)h(x).$$

PROOF. In virtue of (18.13) and (18.19), we have that

$$|f(x, u)| \leqslant |f(x, 0)| + |a(x)u| + b + aM_2^{-1}[M_1(ku)] \qquad (-\infty < u < \infty).$$

By hypothesis, $f(x, 0) \in L^*_{M_2}$. In virtue of B), $a(x)u(x) \in L^*_{M_2}$ for every $u(x) \in L^*_{M_1}$. If $\|u\|_{M_1} \leqslant 1/k$, then $R(|u(x)|) \in L^*_{M_2}$, in which connection we have that

$$\begin{aligned}
\|R(|u(x)|)\|_{M_2} = 2a \left\| \frac{1}{2a} R(|u(x)|) \right\|_{M_2} \leqslant \\
\leqslant 2a \left\{ 1 + \int_G M_2 \left[\frac{1}{2a} R(|u(x)|) \right] dx \right\} \leqslant \\
\leqslant 2a + aM_2\left(\frac{b}{a}\right) \operatorname{mes} G + a \int_G M_1[ku(x)]\, dx \leqslant \\
\leqslant 3a + aM_2\left(\frac{b}{a}\right) \operatorname{mes} G = \beta. \qquad (18.20)
\end{aligned}$$

Consequently, the operator $\boldsymbol{f}$ acts from the sphere $T(\theta, 1/k; L^*_{M_1})$ into the space $L^*_{M_2}$.

To prove the differentiability of the operator $\boldsymbol{f}$ at the point θ we must show that

$$\lim_{\|u\|_{M_1} \to 0} \frac{\|R(|u(x)|)\|_{M_2}}{\|u\|_{M_1}} = 0. \qquad (18.21)$$

Let $\varepsilon > 0$ be given. Since $R'(0) = 0$, a $c_1 > 0$ can be found such that

$$R(|u|) \leqslant \varepsilon |u| \qquad (18.22)$$

for $|u| < c_1$. To each function $u(x) \in L^*_M$ we assign the function $\tilde{u}(x)$ defined by means of the equality

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } |u(x)| \leqslant c_1, \\ 0 & \text{if } |u(x)| > c_1. \end{cases}$$

In virtue of (18.22) and (18.18), we have that

$$\|R(|\tilde{u}(x)|)\|_{M_2} \leqslant \varepsilon \|\tilde{u}\|_{M_2} \leqslant \varepsilon \|u\|_{M_2} \leqslant \varepsilon q \|u\|_{M_1}. \tag{18.23}$$

We can assume, without loss of generality, that the constant u_0 appearing in condition A) is greater than c_1. It follows from condition A) that, for $u \geqslant c_1$ and $\gamma \leqslant c_1/u_0\nu$,

$$R(u) \leqslant R\left(\frac{u_0}{c_1} u\right) \leqslant \mu \frac{R\left(\dfrac{u}{\gamma}\right)}{P\left(\dfrac{u}{\gamma}\right)} P\left(\frac{u_0}{c_1} u\right)$$

and, in virtue of (18.14), that

$$R(u) \leqslant C\mu R\left(\frac{u}{\gamma}\right) P\left(\frac{u_0\gamma}{c_1}\right). \tag{18.24}$$

Let $\|u\|_{M_1} < c_1/(u_0\nu k)$. We set $\gamma = k\|u\|_{M_1}$ in (18.24). Then

$$R(|u(x) - \tilde{u}(x)|) \leqslant C\mu R\left(\frac{|u(x) - \tilde{u}(x)|}{k\|u\|_{M_1}}\right) P\left(\frac{ku_0\|u\|_{M_1}}{c_1}\right),$$

from which, in virtue of (18.20), it follows that

$$\|R(u - \tilde{u})\|_{M_2} \leqslant C\mu\beta P\left(\frac{ku_0\|u\|_{M_1}}{c_1}\right),$$

and, since $\lim\limits_{u\to 0} P(u)/u = 0$, we have that

$$\lim_{\|u\|_{M_1}\to 0} \frac{\|R(u - \tilde{u})\|_{M_2}}{\|u\|_{M_1}} = 0.$$

From the last relation and (18.23) it follows that

$$\overline{\lim_{\|u\|_{M_1}\to 0}} \frac{\|R(u)\|_{M_2}}{\|u\|_{M_1}} \leqslant \overline{\lim_{\|u\|_{M_1}\to 0}} \frac{\|R(\tilde{u})\|_{M_2}}{\|u\|_{M_1}} + \\ + \lim_{\|u\|_{M_1}\to 0} \frac{\|R(u - \tilde{u})\|_{M_2}}{\|u\|_{M_1}} \leqslant \varepsilon q.$$

Since ε is arbitrary, equality (18.21) is valid. ★

As an example, we consider operator (18.12) as an operator in $\{L^4 \to L^2\}$.

The function $f(x, u) = \sin(e^{u^2} - 1)$ satisfies inequality (18.13) with the function $R(u) = eu^2$. Therefore condition A) is satisfied in which we can set $P(u) = u^2$. Since $M_1(u) = u^4$ and $M_2(u) = u^2$, condition (18.17) is satisfied with the function $Q(u) = u^2$. Finally, condition B) is satisfied inasmuch as $f'_u(x, 0) \equiv 0$ in the case under consideration. The validity of condition (18.19) is obvious.

It thus follows from Theorem 18.4 that the operator (18.12) in $\{L^4 \to L^2\}$ is differentiable at the zero θ of the space L^4.

7. *Auxiliary lemma.* We shall need an operator defined by the function $p(u)$, which is the derivative of an N-function $M(u)$.

LEMMA 18.2. *Let $M(u)$ and $N(v)$ be mutually complementary N-functions the second of which satisfies the Δ_2-condition. Suppose that the derivative $p(u)$ of the function $M(u)$ is continuous. Then the operator $\boldsymbol{p}$, defined by means of the equality $\boldsymbol{p}u(x) = p(|u(x)|)$, acts from $\Pi(E_M, 1)$ into $L^*_N = L_N$ and is continuous.*

PROOF. In virtue of Lemma 9.1, the operator $\boldsymbol{p}$ acts from $T(\theta, 1; L^*_M)$ into L_N. It then follows from Theorem 17.2 that the operator $\boldsymbol{p}$ acts from $\Pi(E_M, 1)$ (and, by the same token, from E_M) into L_N. The continuity of the operator $\boldsymbol{p}$ follows from Theorem 17.3. ★

8. *The Gâteaux gradient.* We shall say that the functional $\boldsymbol{F}(u)$, defined on the Banach space E, is *differentiable in the Gâteaux sense*, or *Gâteaux-differentiable, at the point $u \in E$* if, for arbitrary $h \in E$, the function $\boldsymbol{F}(u + th)$ is differentiable with respect to t and the derivative of this function, for $t = 0$, has the form

$$\frac{d}{dt}\boldsymbol{F}(u + th)|_{t=0} = (v, h),$$

where the element v from the conjugate space $\bar{E}$ to the space E does not depend on h. The element v will be called the *Gâteaux gradient* of the functional $\boldsymbol{F}(u)$ at the point u. The operator $\boldsymbol{\Gamma}$, defined by the formula $\boldsymbol{\Gamma}u = v$ on all elements at which $\boldsymbol{F}(u)$ is Gâteaux-differentiable, will also be called the Gâteaux gradient. Clearly, the Gâteaux gradient acts from E into the conjugate space $\bar{E}$. The following assertion holds (see, e.g., VAĬNBERG [1]).

LEMMA 18.3. *Suppose the functional $\boldsymbol{F}(u)$ is Gâteaux-differentiable on some sphere T of the space E and that its Gâteaux gradient is a continuous operator. Then the functional $\boldsymbol{F}(u)$ is differentiable in the*

usual sense and its gradient (*see subsection* 1) *coincides with the Gâteaux gradient.*

PROOF. Let $u \in T$, $u + h \in T$. By definition,

$$\frac{d}{dt} F(u + th) = (\Gamma(u + th), h) \quad (0 \leqslant t \leqslant 1).$$

Integrating this equality, we obtain that

$$F(u + h) - F(u) = \int_0^1 (\Gamma(u + th), h)\, dt.$$

Therefore,

$$|F(u + h) - F(u) - (\Gamma u, h)| = \left| \int_0^1 (\Gamma(u + th) - \Gamma u, h)\, dt \right| \leqslant$$

$$\leqslant \|h\|_E \int_0^1 \|\Gamma(u + th) - \Gamma u\|_{\overline{E}}\, dt.$$

It follows from the continuity of the operator Γ that

$$\lim_{\|h\|_E \to 0} \frac{|F(u + h) - F(u) - (\Gamma u, h)|}{\|h\|_E} = 0. \quad \bigstar$$

9. *Gradient of the Luxemburg norm.* Suppose $M(u)$ and $N(v)$ are mutually complementary N-functions the second of which satisfies the Δ_2-condition. It is assumed everywhere below that the function $p(u) = M'(u)$ is continuous. It will be convenient for us to consider the function $p(u)$ also for negative values of the argument. Clearly, $p(-u) = -p(u)$.

Let $u(x)$, $h(x) \in E_M$. Then the function

$$\varphi(t, k) = \int_G M\left[\frac{u(x) + th(x)}{k}\right] dx \tag{18.25}$$

is defined for all t and $k \neq 0$.

It is easily seen that

$$\frac{\partial \varphi(t, k)}{\partial t} = \frac{1}{k} \int_G p\left[\frac{u(x) + th(x)}{k}\right] h(x)\, dx \tag{18.26}$$

and that

$$\frac{\partial \varphi(t, k)}{\partial k} = -\frac{1}{k^2} \int_G p\left[\frac{u(x) + th(x)}{k}\right][u(x) + th(x)]\,dx. \qquad (18.27)$$

(The legitimacy of differentiating under the integral sign is proved here and in the sequel by means of standard lines of reasoning.)

In virtue of Lemma 18.2, each of the derivatives found, above, is continuous in all arguments simultaneously.

Let $u(x) \in L_M^*$ be a function for which

$$\int_G M\left[\frac{u(x)}{k}\right] dx = 1 \qquad (18.28)$$

for some $k > 0$. We recall (see p. 78) that in this case the number k coincides with the Luxemburg norm: $k = \|u\|_{(M)}$.

Clearly, the Luxemburg norm can be defined with the aid of equality (18.28) for all functions $u(x) \in E_M$ ($\|u\|_{(M)} \neq 0$). This follows from the fact that the integral appearing in the right member of equality (18.28) is finite for all $k \neq 0$, depends continuously on k, and is such that

$$\lim_{k \to 0} \int_G M\left[\frac{u(x)}{k}\right] dx = \infty, \quad \lim_{k \to \infty} \int_G M\left[\frac{u(x)}{k}\right] dx = 0.$$

THEOREM 18.5. *The Luxemburg norm is a differentiable functional in E_M. The gradient Γ of the Luxemburg norm is defined by means of the formula*

$$\Gamma u(x) = \frac{p\left(\dfrac{u(x)}{\|u\|_{(M)}}\right)}{\displaystyle\int_G p\left(\frac{u(x)}{\|u\|_{(M)}}\right)\frac{u(x)}{\|u\|_{(M)}}\,dx} \quad (u(x) \in E_M). \qquad (18.29)$$

PROOF. We first find the Gâteaux gradient of the Luxemburg norm. To this end, we consider the equality

$$\int_G M\left[\frac{u(x) + th(x)}{k}\right] dx = 1 \quad (u(x), h(x) \in E_M). \qquad (18.30)$$

This equality defines k as an implicit function of t. Since the partial derivatives (18.26) and (18.27) in the left member of equality (18.30) are continuous and

$$\frac{\partial \varphi(0, k)}{\partial k} = -\frac{1}{k^2} \int_G p\left(\frac{|u(x)|}{k}\right) |u(x)|\, dx < 0 \quad (\|u\|_{(M)} \neq 0),$$

we have, on the basis of the implicit function theorem (see, e.g. FIHTENGOL'C [1]), that

$$\frac{dk(0)}{dt} = (v, h),$$

where

$$v = \frac{p\left(\dfrac{u(x)}{\|u\|_{(M)}}\right)}{\displaystyle\int_G p\left(\frac{u(x)}{\|u\|_{(M)}}\right) \frac{u(x)}{\|u\|_{(M)}}\, dx}.$$

We have shown that formula (18.29) defines the Gâteaux gradient. Since it follows from Lemma 18.2 that this Gâteaux gradient is a continuous operator, it is the ordinary gradient — in virtue of Lemma 18.3. ★

10. *Gradient of the Orlicz norm.* The N-functions $M(u)$ and $N(v)$ considered in this subsection satisfy the same restrictions as in the preceding subsection. In virtue of Lemma 18.2, for every function $u(x) \in E_M$, the function

$$J(k) = \int_G N[p(k|u(x)|)]\, dx$$

is defined for all values of k and is continuous. Since $J(0) = 0$, $J(\infty) = \infty$, a k^* can be found such that $J(k^*) = 1$. This signifies, in virtue of Lemma 10.4, that the Orlicz norm can be defined with the aid of the equality

$$\|u\|_M = \int_G p(k^*|u(x)|)\, |u(x)|\, dx, \tag{18.31}$$

where

$$\int_G N[p(k^*|u(x)|)]\, dx = 1. \tag{18.32}$$

It will be convenient for us to use, in place of formula (18.32), the equivalent (see (10.7)) equality

$$k^* \int_G |u(x)|\, p(k^*|u(x)|)\, dx - \int_G M[k^*u(x)]\, dx = 1. \qquad (18.33)$$

As was already remarked, the constant k^* is not, generally speaking, uniquely defined. In this subsection, we assume that $p(u)$ does not have intervals of constancy. Then, obviously, k^* is uniquely defined. It is easily verified that k^* is a continuous functional on E_M.

Let $u(x)$, $h(x) \in E_M$. We denote by $k(t)$ the solution of equation (18.33) corresponding to the function $u_t(u) = u(x) + th(x)$.

LEMMA 18.4. *Suppose the function $k(t)$ has the derivative $k'(t)$. Then the Orlicz norm is a differentiable functional on E_M. The gradient Γ of the Orlicz norm is defined by the formula*

$$\Gamma u(x) = p(k^*u(x)) \quad (u(x) \in E_M), \qquad (18.34)$$

where k^* satisfies equality (18.33).

PROOF. Since, in virtue of Lemma 18.2 and the continuity of the functional k^*, the operator Γ defined by formula (18.34) is a continuous operator acting from E_M into L_N, it suffices to prove that Γ is the Gâteaux gradient of the Orlicz norm.

In virtue of (18.31) and (18.32), the Orlicz norm for the function $u(x) \in E_M$ can be defined by the equality

$$\|u\|_M = \frac{1}{k^*}\left(1 + \int_G M[k^*u(x)]\, dx\right),$$

where k^* satisfies equality (18.33). Let $h(x) \in E_M$. Consider the function

$$F(u + th) = \frac{1}{k(t)}\left(1 + \int_G M[k(t)u_t(x)]\, dx\right),$$

where $u_t(x) = u(x) + th(x)$. In virtue of the differentiability of the function $k(t)$ (see remark on p. 188), we have that

$$\frac{d}{dt} F(u + th) = \frac{1}{k^2(t)}\left\{k(t) \int_G p[k(t)u_t(x)][k'(t)u_t(x) - k(t)h(x)]\, dx -\right.$$

$$- k'(t)\left(1 + \int_G M[k(t)u_t(x)]\,dx\right)\Bigg\} =$$

$$= \frac{1}{k^2(t)}\left\{k^2(t)\int_G p[k(t)u_t(x)]h(x)\,dx + \right.$$

$$\left. + k'(t)\left[k(t)\int_G p[k(t)u_t(x)]u_t(x)\,dx - 1 - \int_G M[k(t)u_t(x)]\,dx\right]\right\},$$

from which it follows, in virtue of (18.33), that

$$\frac{d}{dt}\boldsymbol{F}(u + th) = \int_G p[k(t)u_t(x)]h(x)\,dx.$$

Since $k(0) = k^*$ and $u_t(x)|_{t=0} = u(x)$, we have that

$$\frac{d}{dt}\boldsymbol{F}(u + th)|_{t=0} = (v, h),$$

where $v = p(k^*u(x))$. ★

To apply this lemma, one must know under what conditions the function $k(t)$ is differentiable.

LEMMA 18.5. *Suppose the N-function $M(u)$ has a continuous second derivative $p'(u)$ which is positive for $u \neq 0$ and which satisfies the inequality*

$$|up'(u)| \leqslant a + bp(c|u|) \quad (-\infty < u < \infty). \tag{18.35}$$

Then the solution $k(t)$ of equation (18.33) *corresponding to the function $u_t(x) = u(x) + th(x)$ is a differentiable function.*

PROOF. We note first of all that, in virtue of (18.35), the operator $u(x)p'(u(x))$, as also the operator $p(u(x))$, acts from E_M (even from $\Pi(E_M, 1)$) into L_N and it is a continuous operator. Therefore, for an arbitrary pair of functions $u(x)$, $h(x) \in E_M$, with $\|u\|_M \neq 0$, the integrals

$$\int_G u_t^2(x)p'[ku_t(x)]\,dx \text{ and } \int_G u_t(x)h(x)p'[ku_t(x)]\,dx,$$

where $u_t(x) = u(x) + th(x)$, are finite for arbitrary t and $k > 0$ and they are continuous functions of these variables. From this

and Lemma 18.2, it follows that the function

$$\chi(t, k) = k \int_G u_t(x) p[k u_t(x)]\, dx - \int_G M[k u_t(x)]\, dx$$

has the continuous partial derivatives

$$\frac{\partial \chi(t, k)}{\partial t} = k^2 \int_G u_t(x) h(x) p'[k u_t(x)]\, dx$$

and

$$\frac{\partial \chi(t, k)}{\partial k} = k \int_G u_t^2(x) p'[k u_t(x)]\, dx$$

(in this connection, see remark on p. 188).

Since

$$\frac{\partial \chi(0, k)}{\partial k} = k \int_G u^2(x) p'[k u(x)]\, dx > 0,$$

the equation

$$k \int_G u_t(x) p[k u_t(x)] dx - \int_G M[k u_t(x)] dx = 1$$

defines k as an implicit function of t, in which connection the function $k(t)$ has a continuous derivative $k'(t)$. ★

Condition (18.35) is satisfied if the function $p'(u)$ is monotonic. In fact, if $p'(u)$ decreases, then, in virtue of the evenness of $p'(u)$, we have that

$$p(|u|) = \int_0^{|u|} p'(t)\, dt > |u| p'(u).$$

But if $p'(u)$ increases, then

$$p(2|u|) = \int_0^{2|u|} p'(t)\, dt > \int_{|u|}^{2|u|} p'(t)\, dt > |u| p'(u).$$

Lemmas 18.4 and 18.5 imply the next theorem.

THEOREM 18.6. *Suppose the N-function $M(u)$ has a continuous second derivative $p'(u)$ which is positive for $u \neq 0$ and which satisfies inequality* (18.35). *Suppose further that the complementary function $N(v)$ satisfies the Δ_2-condition. Then the Orlicz norm is a differentiable functional on E_M. The gradient Γ of the Orlicz norm is defined by the equality $\Gamma u(x) = p(k^* u(x))$ $(u(x) \in E_M)$, where*

$$\int_G N[p(k^*|u(x)|)]\,dx = 1.$$

Chapter IV

NONLINEAR INTEGRAL EQUATIONS

§ 19. The P. S. Uryson operator

1. *The P. S. Uryson operator.* The operator defined by the formula

$$\boldsymbol{K}u(x) = \int_G k[x, y, u(y)]\,dy \tag{19.1}$$

will be called the *P. S. Uryson operator*. Concerning the function $k(x, y, u)$ we shall assume that it satisfies the Carathéodory conditions, i.e. that it is continuous in u for almost all $x, y \in G$ and it is measurable in both variables x, y, simultaneously, for every u. (We shall not stop to prove the measurability of the sets and functions which are encountered in the subsequent constructions.)

We shall assume that the function $k(x, y, u)$ satisfies the inequality

$$|k(x, y, u)| \leqslant k(x, y)[a(x) + R(|u|)] \tag{19.2}$$

$$(x, y \in G, -\infty < u < \infty),$$

where

$$\int_G\int M[k(x, y)]\,dxdy \leqslant b < \infty, \tag{19.3}$$

$M(u)$ is some N-function, $a(x)$ is a non-negative function, $R(u)$ is a non-negative monotonically increasing function for $u > 0$ which is continuous.

We shall be interested in the problem in what cases are the conditions (19.2) and (19.3) sufficient that the Uryson operator operate in some Orlicz space L_Φ^* and be continuous, bounded, compact, or completely continuous in this space. (An operator is said to be *compact* on the set T if maps every bounded subset $T_1 \subset T$ into a compact set. An operator is said to be *completely continuous* if it is compact and continuous. We note that for nonlinear operators compactness does not imply continuity.)

We shall assume that the conditions (19.2) and (19.3) imply the boundedness of the values of the operator (19.1) on the sphere $T(\theta, r; L^*_\Phi)$:

$$\|\boldsymbol{K}u\|_\Phi \leqslant c \quad (\|u\|_\Phi \leqslant r). \tag{19.4}$$

It is natural to assume that the constant c depends only on the number b and on the functions $a(x)$, $R(u)$ and $M(u)$. Let $k(x)$ be a non-negative function in L^*_M for which

$$\int_G M[k(x)]\,dx \leqslant \frac{b}{\operatorname{mes} G}.$$

Then conditions (19.2) and (19.3) are satisfied for the operator

$$\boldsymbol{K}u(x) = \int_G \frac{k(x) + k(y)}{2} R(|u(y)|)\,dy.$$

Therefore, in virtue of condition (19.4), we have that

$$\left\| \int_G k(y)R(|u(y)|)\,dy + k(x) \int_G R(|u(y)|)\,dy \right\|_\Phi \leqslant 2c \quad (\|u\|_\Phi \leqslant r). \tag{19.5}$$

It follows from this that $k(x) \in L^*_\Phi$. This means that $L^*_M \subset L^*_\Phi$ and, in virtue of Theorem 13.1,

$$\Phi(\alpha u) < M(u) \tag{19.6}$$

for large values of the argument and some $\alpha > 0$.

It also follows from (19.5) that the integrals $\int_G k(y)R(|u(y)|)\,dy$ are finite for arbitrary function $k(x) \in L^*_M$ and, furthermore, that the operator $\boldsymbol{R}u(x) = R(r|u(x)|/2)$ maps the sphere $T(\theta, r; L^*_\Phi)$ into some bounded set in the space L^*_N, where, as usual, $N(v)$ denotes the N-function complementary to $M(u)$. In virtue of Theorem 17.5, positive constants c_1, c_2 and c_3 can be found such that

$$N\left[\frac{1}{c_1} R\left(\frac{ru}{2}\right)\right] \leqslant c_2 + c_3\Phi(u) \quad (-\infty < u < \infty).$$

It follows from this inequality that

$$N[\beta R(\gamma u)] < k\Phi(\alpha u) \tag{19.7}$$

for large values of u.

In combination with the inequalities established above, further investigation is made under the assumption that

$$N[\beta R(\gamma u)] < kM(u) \tag{19.8}$$

for large values of u and that the N-function $\Phi(u)$ satisfies conditions (19.6) and (19.7).

2. *Boundedness of the Uryson operator.* Our main attention will be centered on the case when the function $N(v)$, complementary to $M(u)$, satisfies the Δ'-condition.

LEMMA 19.1. *Suppose the N-function $N(v)$ satisfies the Δ'-condition. Assume that conditions* (19.2), (19.3), (19.6) *and* (19.7) *are satisfied. Finally, let $a(x) \in L_N^* = L_N$. The operator* (19.1) *is defined on the sphere $T(\theta, \gamma/\alpha; L_\Phi^*)$ and the set of its values have uniformly bounded norms in L_Φ^*:*

$$\|\boldsymbol{K}u\|_\Phi \leqslant C\,\|k(x, y)\|_{\hat{M}} \quad \left(\|u\|_\Phi \leqslant \frac{\gamma}{\alpha}\right), \tag{19.9}$$

where the constant C does not depend on the kernel $k(x, y)$.

PROOF. Suppose inequality (19.7) is satisfied for $u \geqslant u_0$. Since

$$\|a(x) + R(|u(x)|)\|_N \leqslant \|a\|_N + \frac{1}{\beta}\|\beta R(|u(x)|)\|_N \leqslant$$

$$\leqslant \|a\|_N + \frac{1}{\beta}\left\{1 + \int_G N[\beta R(|u(x)|)]\,dx\right\},$$

we have, in virtue of (19.7), that for $\|u\|_\Phi \leqslant \gamma/\alpha$

$$\|a(x) + R(|u(x)|)\|_N \leqslant$$

$$\leqslant \|a\|_N + \frac{1}{\beta}\left\{1 + N[\beta R(\gamma u_0)]\ \text{mes}\ G + k\int_G \Phi\left[\frac{\alpha}{\gamma}u(x)\right]dx\right\} \leqslant$$

$$\leqslant \|a\|_N + \frac{1}{\beta}\left\{1 + k + N[\beta R(\gamma u_0)]\ \text{mes}\ G\right\},$$

i.e.

$$\|a(x) + R(|u(x)|)\|_N \leqslant C_1 \quad \left(\|u\|_\Phi \leqslant \frac{\gamma}{\alpha}\right). \tag{19.10}$$

In virtue of Theorem 15.4, the linear integral operator

$$\boldsymbol{A}v(x) = \int_G k(x, y)v(y)\,dy$$

operates from L_N into L^*_Φ where

$$\|\boldsymbol{A}v\|_\Phi \leqslant 2l\|k(x, y)\|_{\hat{M}}\|v\|_N. \tag{19.11}$$

In virtue of (19.2),

$$|\boldsymbol{K}u(x)| \leqslant \boldsymbol{A}[a(x) + R(|u(x)|)],$$

from which it follows, in virtue of (19.11) and (19.10), that

$$\|\boldsymbol{K}u\|_\Phi \leqslant 2l\|k(x, y)\|_{\hat{M}}\|a(x) + R(|u(x)|)\|_N \leqslant 2lC_1\|k(x, y)\|_{\hat{M}}. \bigstar$$

3. *Transition to a simpler operator.* We shall assume that the conditions of Lemma 19.1 are satisfied. In virtue of Theorem 17.1, one can find a sequence of closed sets $\hat{G}_n \subset \hat{G}$ such that mes $(\hat{G} \setminus \hat{G}_n) < 1/n$, the function $k(x, y, u)$ is continuous in all variables on the sets $\hat{G}_n \times (-\infty, \infty)$, and the functions $k(x, y)$ and $k(x, y)a(y)$ are continuous on the sets $\hat{G}_n$. We set

$$k_n(x, y, u) = \begin{cases} k(x, y, u) & \text{if } \{x, y\} \in \hat{G}_n, \\ 0 & \text{if } \{x, y\} \bar{\in} \hat{G}_n. \end{cases}$$

Each of the functions $k_n(x, y, u)$ satisfies condition (19.2) and, in virtue of Lemma 19.1, it determines the Uryson operator

$$\boldsymbol{K}_n u(x) = \int_G k_n[x, y, u(y)]\,dy,$$

acting from the sphere $T(\theta, \gamma/\alpha; L^*_\Phi)$ into L^*_Φ.

Clearly,

$$\boldsymbol{K}u(x) - \boldsymbol{K}_n u(x) = \int_G k[x, y, u(y)]\kappa(x, y; \hat{G} \setminus \hat{G}_n)\,dy.$$

In virtue of (19.2) and Lemma 19.1, we have that

$$\|\boldsymbol{K}u - \boldsymbol{K}_n u\|_{\Phi} \leqslant \left\| \int_G k(x,y)\, \kappa(x, y; \hat{G} \setminus \hat{G}_n)[a(x) + R(|u(y)|)]\, dy \right\|_{\Phi} \leqslant$$

$$\leqslant C \|k(x, y)\kappa(x, y; \hat{G} \setminus \hat{G}_n)\|_{\hat{M}}, \quad \|u\|_{\Phi} \leqslant \frac{\gamma}{\alpha}.$$

We shall assume, in addition, that $k(x, y) \in \hat{E}_M$. In virtue of Theorem 10.3, $\lim\limits_{n\to\infty} \|k(x, y)\kappa(x, y; \hat{G} \setminus \hat{G}_n)\|_{\hat{M}} = 0$. Therefore,

$$\lim_{n\to\infty} \sup_{\|u\|_{\Phi} \leqslant (\gamma/\alpha)} \|\boldsymbol{K}u - \boldsymbol{K}_n u\|_{\Phi} = 0.$$

We showed that the operator $\boldsymbol{K}$ can be uniformly approximated by the operators $\boldsymbol{K}_n$ if $k(x, y) \in \hat{E}_M$ under the conditions of Lemma 19.1.

Thus, to prove the continuity or the compactness of the operator $\boldsymbol{K}$, it suffices to prove the continuity or the compactness of the operators $\boldsymbol{K}_n$. The operators $\boldsymbol{K}_n$ are defined by the functions $k_n(x, y, u)$ for which the inequalities

$$|k_n(x, y, u)| \leqslant a_n + b_n R(|u|), \quad -\infty < u < \infty,$$

are valid, where

$$a_n = \max_{\{x,y\} \in \hat{G}_n} |k(x, y)a(x)|, \quad b_n = \max_{\{x,y\} \in \hat{G}_n} |k(x, y)|.$$

Thus, the continuity and compactness of an arbitrary operator $\boldsymbol{K}$ under the assumptions made above will be proved if the corresponding properties are proved for the Uryson operator

$$\boldsymbol{K}u(x) = \int_G k[x, y, u(y)]\, dy$$

under the assumption that

$$|k(x, y, u)| \leqslant a + R(|u|); \; x, y \in G, \; -\infty < u < \infty. \tag{19.12}$$

4. *A second transition to a simpler operator.* Suppose condition (19.12) is satisfied. We define a new sequence of functions $k_n(x, y, u)$

by means of the equality

$$k_n(x, y, u) = \begin{cases} k(x, y, u) & \text{if } |u| \leqslant n, \\ k(x, y, n)(n + 1 - u) & \text{if } n < u < n + 1, \\ k(x, y, -n)(u + n - 1) & \text{if } -n - 1 < u < -n, \\ 0 & \text{if } |u| \geqslant n + 1. \end{cases}$$

Suppose the Uryson operators $\boldsymbol{K}_n$ are defined by the functions $k_n(x, y, u)$. We shall show that

$$\lim_{n\to\infty} \sup_{\|u\|_{\Phi} \leqslant (\gamma/\alpha)} \|\boldsymbol{K}u - \boldsymbol{K}_n u\|_{\Phi} = 0. \tag{19.13}$$

Let $u(x) \in T(\theta, \gamma/\alpha; L^*_{\Phi})$. We denote the set $G\{|u(x)| > n\}$ by G'. Clearly,

$$\operatorname{mes} G' \leqslant \frac{1}{\Phi\left(\frac{\alpha n}{\gamma}\right)} \int_G \Phi\left[\frac{\alpha u(x)}{\gamma}\right] dx \leqslant$$

$$\leqslant \frac{1}{\Phi\left(\frac{\alpha n}{\gamma}\right)} \left\| \frac{\alpha}{\gamma} u \right\|_{\Phi} \leqslant \frac{1}{\Phi\left(\frac{\alpha n}{\gamma}\right)}.$$

We shall estimate $\|\boldsymbol{K}u - \boldsymbol{K}_n u\|_{\Phi}$. It follows from the definition of the operators $\boldsymbol{K}_n$ that

$$|\boldsymbol{K}u(x) - \boldsymbol{K}_n u(x)| \leqslant \int_G |k[x, y, u(y)] - k_n[x, y, u(y)]|\, dy \leqslant$$

$$\leqslant \int_{G'} |k[x, y, u(y)]|\, dy + \int_{G'} |k[x, y, \psi(y)]|\, dy,$$

where $\psi(x) = n\kappa(x; G')\operatorname{sgn} u(x)$, $\|\psi\|_{\Phi} \leqslant \|u\|_{\Phi} \leqslant \gamma/\alpha$.

In virtue of (19.12), we have that

$$|\boldsymbol{K}u(x) - \boldsymbol{K}_n u(x)| \leqslant \int_G \kappa(x, y; \hat{G}')[a + R(|u(y)|)]\, dy +$$

$$+ \int_G \kappa(x, y; \hat{G}')[a + R(|\psi(y)|)]\, dy,$$

where $\hat{G}' = G \times G'$. Lemma 19.1 implies the inequality

$$\|\boldsymbol{K}u - \boldsymbol{K}_n u\|_{\Phi} \leqslant 2C \|\kappa(x, y; \hat{G}')\|_{\hat{M}} = 2C \operatorname{mes} \hat{G}' N^{-1}\left(\frac{1}{\operatorname{mes} \hat{G}'}\right).$$

Making use of the estimate for mes G', we obtain the inequality

$$\|\boldsymbol{K}u - \boldsymbol{K}_n u\|_{\Phi} \leqslant \frac{2C \operatorname{mes} G}{\Phi\left(\frac{\alpha n}{\gamma}\right)} N^{-1}\left[\frac{\Phi\left(\frac{\alpha n}{\gamma}\right)}{\operatorname{mes} G}\right], \quad \|u\|_{\Phi} \leqslant \gamma/\alpha,$$

from which 19.13 does indeed follow.

Thus, the continuity and compactness of the Uryson operator under the conditions of Lemma 18.1 with the additional condition that $k(x, y) \in \hat{E}_M$ will be proved if we prove the corresponding properties of Uryson operators with bounded function $k(x, y, u)$:

$$|k(x, y, u)| \leqslant d \quad (x, y \in G, \ -\infty < u < \infty), \tag{19.14}$$

satisfying the condition

$$k(x, y, u) \equiv 0, \quad |u| \geqslant u_0, \tag{19.15}$$

where u_0 is a positive number.

5. *A third transition to a simpler operator.* Suppose the conditions (19.14) and (19.15) are satisfied. In virtue of Theorem 17.1, there exists a sequence of closed sets $\hat{G}_n \subset \hat{G}$ such that mes $(\hat{G} \setminus \hat{G}_n) = 0$ and the function $k(x, y, u)$ is continuous in all variables $\{x, y\} \in \hat{G}_n$, $-\infty < u < \infty$. By Uryson's theorem, there exist functions $k_n(x, y, u)$, which are continuous in all variables, coincide with $k(x, y, u)$ for $\{x, y\} \in \hat{G}_n$, and satisfy the conditions $|k_n(x, y, u)| \leqslant d$ for $x, y \in G$, $-\infty < u < \infty$, $k_n(x, y, u) \equiv 0$ for $|u| \geqslant u_0$.

We now consider the Uryson operators $\boldsymbol{K}_n$ defined by the functions $k_n(s, y, u)$. Each of the operators $\boldsymbol{K}_n$ maps an arbitrary Orlicz space into a uniformly bounded, equi-continuous family of functions. Therefore, the set of values of every operator $\boldsymbol{K}_n$ is compact in C and, a fortiori, in an arbitrary Orlicz space.

Suppose the sequence of functions $u_i(x)$ $(i = 1, 2, \cdots)$ converges to the function $u_0(x)$ with respect to the Orlicz space norm. Then this sequence also converges to the function $u_0(x)$ in measure.

The operator $\boldsymbol{K}_n$ maps every sequence of functions which converges in measure to $u_0(x)$ into a sequence of functions which converges for every x, which fact follows from the possibility of passing to the limit under the integral sign:

$$\boldsymbol{K}_n u_0(x) = \int_G k_n[x, y, u_0(y)]\,dy = \\ = \lim_{i\to\infty} \int_G k_n[x, y, u_i(y)]\,dy = \lim_{i\to\infty} \boldsymbol{K}_n u_i(x).$$

The sequence $\boldsymbol{K}_n u_i(x)$ converges uniformly to the functions $\boldsymbol{K}_n u_0(x)$ inasmuch as it is compact. Thus, $\boldsymbol{K}_n \in \{L^*_{\Phi} \to C\,;$ comp. c.$\}$ and, a fortiori, $\boldsymbol{K}_n \in \{L^*_{\Phi} \to L^*_{\Phi}\,;$ comp. c.$\}$.

For any function $u(x) \in L^*_{\Phi}$, the obvious inequality

$$|\boldsymbol{K}u(x) - \boldsymbol{K}_n u(x)| \leqslant 2d \int_G \kappa(x, y\,;\, \hat{G} \setminus \hat{G}_n)\,dy$$

is valid, from which it follows that

$$\|\boldsymbol{K}u - \boldsymbol{K}_n u\|_{\Phi} \leqslant C_1 \|\kappa(x, y\,;\, \hat{G} \setminus \hat{G}_n)\|_{\hat{\Phi}}.$$

The inequality obtained signifies that the continuous and compact operators $\boldsymbol{K}_n$ converge uniformly to the operator $\boldsymbol{K}$ on the entire space L^*_{Φ}. This means that the operator $\boldsymbol{K}$ is continuous and compact.

6. *Fundamental theorem on the complete continuity of Uryson's operator.* We now formulate the result obtained in the preceding subsections.

THEOREM 19.1. *Suppose $M(u)$ and $N(v)$ are mutually complementary N-functions the second of which satisfies the Δ'-condition. Let*

$$|k(x, y, u)| \leqslant k(x, y)[a(x) + R(|u|)]\,; \qquad (19.16)$$

$$x, y \in G, \quad -\infty < u < \infty,$$

where $k(x, y) \in \hat{E}_M$, $a(x) \in L^*_N$, and $R(u)$ is a non-negative non-decreasing function. Finally, suppose positive numbers β, γ and K can be found such that

$$N[\beta R(\gamma u)] \leqslant KM(u) \qquad (19.17)$$

for large values of the argument. Then the operator

$$\boldsymbol{K}u(x) = \int_G k[x, y, u(y)]\,dy \tag{19.18}$$

belongs to $\{T(\theta, \gamma; L^*_\Phi) \to L^*_\Phi;$ comp. c.$\}$, *where* $\Phi(u)$ *is any N-function which satisfies the inequalities*

$$N[\beta R(\gamma u)] \leqslant K\Phi(u) \leqslant KM(u) \tag{19.19}$$

for large values of the argument.

The natural question arises under what additional assumptions is the operator $\boldsymbol{K}$ defined not only on the sphere $T(\theta, \gamma; L^*_\Phi)$ but also on the entire space L^*_Φ? This will occur if arbitrarily large γ's can be taken in conditions (19.17) and (19.19).

In particular, the operator $\boldsymbol{K}$ acts in all L_Φ if the N-function $\Phi(u)$ satisfies the Δ_2-condition since, in this case,

$$N[\beta R(2^s \gamma u)] \leqslant K\Phi(2^s u) \leqslant K_1\Phi(u) \leqslant K_1 M(u)$$

for large values of the argument.

It is verified analogously that the Uryson operator is completely continuous in the entire space L^*_Φ if the function $R(u)$ satisfies the Δ_2-condition: $R(2u) \leqslant K_1 R(u)$ for large values of the argument.

Conditions (19.17) and (19.19) can, in some cases, be rewritten in a simpler form.

Let us assume that the function $\Phi(u)$ is such that $N[\Phi(u)] \sim \Phi(u)$, i.e. that $N[\Phi(u)] \leqslant \Phi(\alpha u)$ for large values of the argument. Then conditions (19.17) and (19.19) are satisfied if

$$R(\alpha\gamma u) \leqslant K\Phi(u) \leqslant KM(u). \tag{19.20}$$

This follows from the obvious inequalities

$$N[R(\gamma u)] \leqslant N\left[K\Phi\left(\frac{u}{\alpha}\right)\right] \leqslant K_1 N\left[\Phi\left(\frac{u}{\alpha}\right)\right] \leqslant K_1\Phi(u) \leqslant K_1 M(u).$$

Various conditions for the complete continuity of nonlinear integral operators in the spaces L^α are given in KRASNOSEL'SKIĬ [5]. All these theorems are concerned with the case when nonlinearities of polynomial type are investigated. The fundamental theorems, given in KRASNOSEL'SKIĬ [5], are comprised in Theorem 19.1, proved above. However, Theorem 19.1 also gives conditions for complete

continuity in certain Orlicz spaces for integral operators which contain essentially non-power, e.g. exponential, non-linearities.

For example, let

$$|k(x, y, u)| \leqslant k(x, y)e^{\alpha|u|}; \quad x, y \in G, \ -\infty < u < \infty, \tag{19.21}$$

with

$$|k(x, y)| \leqslant a|\ln r|^{1-\beta_0} + b, \tag{19.22}$$

where r is the distance between the points $x, y \in G$.

Let p.p. $\Phi(u) = e^{|u|^{1+\beta}}$, where $0 \leqslant \beta \leqslant \beta_0$. Let $M(u) = \Phi(u)$. Clearly, $k(x, y) \in \hat{E}_M$ since

$$\int_G \int \exp |\lambda k(x, y)|^{1+\beta} dx\, dy < \infty$$

for arbitrary $\lambda > 0$.

Since $M(u)$ satisfies the Δ^2-condition, the function $N(v)$ complementary to it satisfies the Δ'-condition. In virtue of Theorem 6.3, we have that $N[\Phi(u)] \sim \Phi(u)$. Therefore, we can use Theorem 19.1, in which conditions (19.7) and (19.9) are replaced by condition (19.20).

Condition (19.20) is satisfied and, furthermore, it is satisfied for arbitrary $\gamma > 0$ if $\beta > 0$.

Thus, if conditions (19.21) and (19.22) are satisfied, then the Uryson operator is completely continuous in some sphere of the space L^*_{Φ}, where $\Phi_0(u) = e^{|u|} - |u| - 1$ and it is completely continuous in the entire space $L^*_{\Phi_0}$, where $\Phi_0(u)$ is an arbitrary function of the form $\Phi(u) = e^{|u|^{1+\beta}} - 1$, with $0 < \beta < \beta_0$.

An analogous line of reasoning shows that the Uryson operator is completely continuous in the entire space $L^*_{\Phi_0}$, where $\Phi_0(u) = (e^{|u|^{1-\beta_0}} - 1)|u|$ if

$$|k(x, y, u)| \leqslant (a|\ln r| + b)e^{|u|^{1-\beta}}, \tag{19.23}$$

with $\beta_0 < \beta < 1$.

7. *The case of weak nonlinearities.* In the case considered above, the function $M(u)$ increased more rapidly than some power function inasmuch as the N-function $N(v)$ complementary to it satisfied the Δ'-condition. This signifies that for the cases studied the function $k(x, y)$ in condition (19.2) belongs to some L^{α} ($\alpha > 1$). In the present subsection, we shall asume

that

$$\int_G \int M[k(x, y)]\,dx\,dy < \infty, \tag{19.24}$$

where $M(u) \prec |u|^\alpha$ for all $\alpha > 1$. Then the N-function $N(v)$ complementary to it increases more rapidly than any power function. We shall assume that $N(v)$ satisfies the Δ_3-condition. In this connection, the function $M(u)$ itself satisfies the Δ_2-condition.

As was elucidated above, in the study of Uryson operators, it is natural to assume that condition (19.8) is satisfied for large values of the argument:

$$N[\beta R(\gamma u)] < KM(u) < M(Ku). \tag{19.25}$$

In virtue of Theorem 6.3, the inequality $N^{-1}[M(u)] < K_1 N^{-1}(u)$ is valid for large values of the argument. It therefore follows from (19.25) that $\beta R(\gamma u) < N^{-1}[M(Ku)] < K_1 N^{-1}(Ku)$ for large values of u.

In virtue of Theorem 6.1,

$$\beta R(\gamma u) < \frac{K_1}{K} \frac{KuN^{-1}(Ku)}{u} < \frac{K^1}{Ku} M(K_2 u)$$

for large values of u. Since the N-function $M(u)$ satisfies the Δ_2-condition, it follows from (19.25) finally that

$$R(u) < C \frac{M(u)}{u} \tag{19.26}$$

for large values of u.

We note that condition (19.25) follows from (19.26) and, furthermore, (19.26) implies the validity, for large values of u and certain β and γ, of the inequality

$$N[\beta R(u)] < \frac{1}{\gamma} u. \tag{19.27}$$

It follows from the last inequality that $R(u) < C_1 N^{-1}(u)$ for large values of the argument. This means that in the case under consideration the nonlinearity of $R(u)$ must be "very" weak. In fact, it follows from the fact that $N(v)$ satisfies the Δ_3-condition that $N(v)$ increases more rapidly than any power function. There-

fore the inequalities obtained signify that $R(u)$ increases more slowly than any power function $|u|^{\varepsilon}$, where $\varepsilon > 0$, If $N(v)$ satisfies the Δ^2-condition, then, as was noted above (on p. 43), $N(v)$ increases more rapidly than some function $e^{v^{\alpha}}$ $(\alpha > 0)$.

From this it follows that, in the case under consideration, $R(u)$ increases more slowly than $(\ln u)^{1/\alpha}$.

THEOREM 19.2. *Suppose $M(u)$ and $N(v)$ are mutually complementary N-functions the second of which satisfies the Δ_3-condition. Let*

$$|k(x, y, u)| \leqslant k(x, y)[a(x) + R(|u|)]; \tag{19.28}$$

$$x, y \in G, \ -\infty < u < \infty,$$

where $k(x, y) \in \hat{L}^{*}_{M} = \hat{L}_{M}$, $a(x) \in L^{*}_{N}$, and $R(u)$ is a non-negative function which does not decrease for $u > 0$. Finally, suppose a $C > 0$ can be found such that inequality (19.26) *is satisfied for large values of the argument. Then there exists an Orlicz space L^{*}_{Φ} in which the completely continuous operator*

$$\boldsymbol{K}u(x) = \int_{G} k[x, y, u(y)]\,dy \tag{19.29}$$

acts.

PROOF. Since the function $M[k(x, y)]$ is summable on $\hat{G}$, an N-function $\Phi(u)$, satisfying the Δ_2-condition (and even the Δ'-condition), can be found (see p. 61) such that

$$\int_{\hat{G}}\int \Phi\{M[k(x, y)]\}\,dx\,dy < \infty. \tag{19.30}$$

We shall show that the operator (19.29) acts in the space L^{*}_{Φ}. The validity of inequality (19.27) for large values of u follows from inequality (19.26). Suppose it is satisfied for $u \geqslant u_0$. Let $u(x) \in L^{*}_{\Phi} = L_{\Phi}$. Since

$$\|a(x) + R(|u(x)|)\|_{N} \leqslant \|a\|_{N} + \frac{1}{\beta}\|\beta R(|u(x)|)\|_{N} \leqslant$$

$$\leqslant \|a\|_{N} + \frac{1}{\beta}\left\{1 + \int_{G} N[\beta R(|u(x)|)]\,dx\right\},$$

we have, in virtue of (19.27), that

$$\|a(x) + R(|u(x)|)\|_N \leqslant$$

$$\leqslant \|a\|_N + \frac{1}{\beta}\left\{1 + N[\beta R(u_0)] \text{ mes } G + \frac{1}{\gamma}\int\limits_G |u(x)|\,dx\right\} \leqslant$$

$$\leqslant \|a\|_N + \frac{1}{\beta}\{1 + N[\beta R(u_0)] \text{ mes } G + \mu\|u\|_\Phi\},$$

where μ is some constant.

Thus, for $\|u\|_\Phi \leqslant r$, we have that

$$\|a(x) + R(|u(x)|)\|_N \leqslant C(r). \tag{19.31}$$

Applying to the linear integral operator

$$Av(x) = \int\limits_G k(x, y)v(y)\,dy$$

condition a) of Theorem 15.4 (setting $M_1(u) = N(u)$, $M_2(u) = \Phi(u)$, $\Psi(u) = \Phi[M(u)]$), we convince ourselves, in virtue of (19.30), that the operator A acts from the space L_N^* into the space L_Φ and is continuous, with

$$\|Av\|_\Phi \leqslant 2l\|k(x, y)\|_{\hat{\Psi}}\|v\|_N. \tag{19.32}$$

In virtue of (19.28), we have that

$$|Ku(x)| \leqslant A[a(x) + R(|u(x)|)],$$

from which it follows, in virtue of (19.31) and (19.32), that

$$\|Ku\|_\Phi \leqslant 2l\|k(x, y)\|_{\hat{\Psi}}\|a(x) + R(|u(x)|)\|_N \leqslant$$

$$\leqslant 2lC(r)\,\|k(x, y)\|_{\hat{\Psi}}.$$

The continuity and compactness of the operator (19.29) are proved the same way that Theorem 19.1 was proved. ★

We now consider a simple example. Suppose

$$|k(x, y, u)| \leqslant k(x, y)[a + \ln(|u| + 1)] \tag{19.33}$$

and

$$\int\limits_G\int |k(x, y)| \ln(|k(x, y)| + 1)\,dx\,dy < \infty. \tag{19.34}$$

Then the operator (19.29) acts in some Orlicz space and is completely continuous there. To prove this, we must apply Theorem 19.2 in which the function

$$M(u) = (1 + |u|) \ln (1 + |u|) - |u|.$$

8. *Hammerstein operators.* We now consider Uryson operators of the special form

$$\boldsymbol{K}u(x) = \int_G k(x, y) f[y, u(y)]\, dy. \tag{19.35}$$

Such operators are called *Hammerstein operators.*

The conditions found above under which the Uryson operators act in some Orlicz space and are completely continuous there are, naturally, applicable also in the investigation of the operator (19.33). However, another method can be utilized in the study of this operator in certain cases.

Suppose E_1 and E_2 are two Banach spaces. We shall assume that the operator $\boldsymbol{f}$: $\boldsymbol{f}u(x) = f[x, u(x)]$, acts from some sphere $T \subset E_1$ into the space E_2, is continuous and is bounded on this sphere. Suppose the linear integral operator

$$\boldsymbol{A}v(x) = \int_G k(x, y) v(y)\, dy$$

acts from E_2 into E_1 and is continuous. Since the operator (19.35) can be represented in the form of the composition $\boldsymbol{K} = \boldsymbol{A}\boldsymbol{f}$, under the indicated conditions, it obviously acts from the sphere T into E_1, is continuous and bounded. If the operator $\boldsymbol{A}$ is completely continuous, then the operator (19.35) is also completely continuous.

E_1 and E_2 can be considered to be two Orlicz spaces. In § 17, we found conditions for the continuity and boundedness of the operator $\boldsymbol{f}$. Combination of these conditions with the conditions for continuity (§ 15) and complete continuity (§ 16) of the operator $\boldsymbol{A}$ yields sufficient conditions for the continuity and complete continuity of the Hammerstein operator.

§ 20. Some existence theorems

1. *Problems under consideration.* Suppose A is an operator, which, generally speaking, is nonlinear and acts in some Banach space E. We shall point out some problems which arise in the consideration of the equation

$$A\varphi = \lambda\varphi. \tag{20.1}$$

The first problem is to find conditions under which equation (20.1) has solutions for fixed values of λ. As a rule, it is desirable to supplement the conditions for the existence of solutions by the conditions for uniqueness of the solution.

In many cases the operator A possesses the property that $A\theta = \theta$, where θ is the zero of the space E. Then equation (20.1) has the trivial, zero solution for all values of the numerical parameter λ. In these cases, solutions different from the trivial solution are of interest. Such solutions exist only for isolated values of the parameter λ. It is customary to call the non-zero solutions of equation (20.1) *characteristic vectors* (or *characteristic functions*) of the operator A. The numbers λ for which equation (20.1) has non-zero solutions are called *characteristic values* of the operator A.

The second problem is to find conditions under which the operator A has characteristic vectors.

The totality of characteristic values of the nonlinear operator A is called its *spectrum* (in analogy with the linear operator case). If the spectrum of an operator fills an interval, it follows from this that the operator has a continuum of characteristic vectors. There can be cases when an infinite (countable or continuous) set of characteristic functions corresponds to one characteristic value.

The third problem is the investigation of the spectrum of a nonlinear operator and the study of the topological structure of the set of characteristic vectors.

It turns out that under rather general assumptions the sets of characteristic vectors are entities of the same type as continuous curves; these entities are called continuous branches. We now introduce the corresponding definition. A set $\mathfrak{N} \subset E$ is called a *continuous branch* in the spherical layer $a < \|u - u_0\| < b$ if the intersection of the set $\mathfrak{N}$ with the boundary S of any region $\mathfrak{M}$ which contains the sphere $\|u - u_0\| \leqslant a$ and is contained with its boundary in the sphere $\|u - u_0\| < b$ is non-void.

Of basic interest are the conditions under which a nonlinear operator has characteristic vectors with arbitrarily small norms. Let λ_0 be some number and suppose to every $\varepsilon > 0$ there corresponds a λ such that $|\lambda - \lambda_0| < \varepsilon$ and that for this value of λ equation (20.1) has at least one non-zero solution φ satisfying the condition $\|\varphi\| < \varepsilon$. Then the number λ_0 is called a *branch point* of the nonlinear operator $\boldsymbol{A}$.

The fourth problem is the investigation of branch points.

To solve the problems enumerated above (and a number of others which we did not mention) qualitative methods of nonlinear functional analysis are being worked out at the present time. The application of general propositions to the investigation of concrete equations (20.1) requires that the operator $\boldsymbol{A}$ possess definite "good" properties: that it be continuous and bounded, in other cases that it be completely continuous, that it be differentiable, that it be the gradient of some functional, and so on.

In connection with this, the application of general theorems of nonlinear functional analysis to the study of concrete nonlinear integral equations requires the construction of a functional space in which the integral operator acts and possesses certain "good" properties.

In the majority of known investigations, the space C of continuous functions and various L^α spaces are taken as the functional space E. This circumstance leads to the fact that various restrictions are placed upon the functions which appear in the equation. The application of Orlicz spaces leads to other (sometimes weaker) restrictions and permits us to consider new classes of equations.

Combination of the results of the preceding sections with the general propositions of nonlinear functional analysis leads to new existence theorems, theorems on characteristic functions and branch points, and so on.

Below, we shall introduce some examples of such a combination. The reader who is familiar with nonlinear functional analysis can easily extend the list of such examples.

2. *The existence of solutions.* One of the most common methods to prove existence theorems consists in utilizing Schauder's fixed-point principle.

SCHAUDER'S PRINCIPLE. *Suppose the completely continuous operator $\boldsymbol{A}$ maps the sphere T of some Banach space B into a subset*

of T. Then the sphere T contains at least one element u_0 such that $u_0 = \boldsymbol{A}u_0$.

We consider the equation

$$u(x) = \lambda \int_G k[x, y, u(y)]\,dy + f_0(x). \tag{20.2}$$

Suppose the conditions (see § 19) are satisfied under which the operator

$$\boldsymbol{K}u(x) = \int_G k[x, y, u(y)]\,dy \tag{20.3}$$

is defined on the sphere $T(\theta, \gamma; L_\Phi^*)$ and is a completely continuous operator with its set of values in L_Φ^*. Let

$$\sup_{\|u\|_\Phi \leqslant \gamma} \|\boldsymbol{K}u\|_\Phi = a.$$

We shall assume that $f_0(x) \in L_\Phi^*$ and that $\|f_0\|_\Phi = \delta < \gamma$. Then, for

$$|\lambda| \leqslant \frac{\gamma - \delta}{a}, \tag{20.4}$$

the operator defined by the right member of equation (20.2) maps the sphere $T(\theta, \gamma; L_\Phi^*)$ into a subset of itself:

$$\|\lambda \boldsymbol{K}u + f_0\|_\Phi \leqslant |\lambda|\, a + \|f_0\|_\Phi \leqslant \gamma \quad (\|u\|_\Phi \leqslant \gamma).$$

The operator $\lambda \boldsymbol{K}u(x) + f_0(x)$ is completely continuous since the operator $\boldsymbol{K}$ is completely continuous.

It thus follows from the Schauder principle that *equation* (20.2) *has at least one solution in the space L_Φ^* for sufficiently small λ.*

If the operator $\boldsymbol{K}$ is defined on the entire space L_Φ^*, then, in making use of the Schauder principle, one can consider spheres of various radii γ. It is natural in this case to consider spheres with radii for which the right member of (20.4) takes on the largest possible value. In particular, equation (20.2) has a solution for all λ if

$$\overline{\lim_{\gamma \to \infty}} \frac{\gamma}{\sup\limits_{\|u\|_\Phi \leqslant \gamma} \|\boldsymbol{K}u\|_\Phi} = \infty, \tag{20.5}$$

since if this condition is satisfied for every λ a sufficiently large γ can be found for which (20.4) is satisfied.

The line of reasoning introduced above enables us to prove, for example, the following assertion: *equation* (20.2) *has a solution for arbitrary* λ *if the conditions of Theorem* 19.2 *are satisfied in which* $N(v)$ *satisfies the* Δ^2*-condition and if* $f_0(x)$ *is a summable function.*

In fact, we choose the N-function $\Phi(u)$, which satisfies the Δ'-condition, so that, on the one hand, inequality (19.30) is satisfied and, on the other hand, $f_0(x) \in L^*_\Phi$. Equation (20.2) can then be considered as an operator equation in the space L^*_Φ.

It follows from the line of reasoning followed in the proof of Theorem 19.2 that

$$\|\boldsymbol{K}u\|_\Phi \leqslant 2l\|k(x, y)\|_{\hat{\Psi}}\|a(x) + R(|u(x)|)\|_N \leqslant$$

$$\leqslant \alpha + \beta\|R(|u(x)|)\|_N \leqslant \alpha_1 + \beta_1\|N^{-1}(|u(x)|)\|_N.$$

Therefore, to prove equality (20.5), it suffices to prove that

$$\lim_{\|u\|_\Phi \to 0} \frac{\|N^{-1}(|u(x)|)\|_N}{\|u\|_\Phi} = 0. \tag{20.6}$$

Since the N-function $N(v)$ satisfies the Δ^2-condition, a u_0 can be found (see (6.12)) such that $N(u)N(v) \leqslant N(uv)$ for $u, v \geqslant u_0$. Setting $N(u) = \rho$, $N(v) = t$ in this inequality and applying the function $N^{-1}(u)$ to both members of the inequality we arrive at the inequality $N^{-1}(\rho t) \leqslant N^{-1}(\rho)N^{-1}(t)$, which is valid for ρ, $t \geqslant \rho_0 = N(u_0)$. If $\rho \geqslant \rho_0$ and $t < \rho_0$, then

$$N^{-1}(\rho t) \leqslant N^{-1}(\rho\rho_0) \leqslant N^{-1}(\rho)N^{-1}(\rho_0).$$

Thus, the inequality

$$N^{-1}(\rho t) \leqslant N^{-1}(\rho)N^{-1}(t) + N^{-1}(\rho)N^{-1}(\rho_0)$$

is valid for $\rho \geqslant \rho_0$ and for all $t > 0$. It follows from this inequality that, for $\rho \geqslant \rho_0$,

$$N^{-1}(|u(x)|) = N^{-1}\left(\rho \frac{u(x)}{\rho}\right) \leqslant$$

$$\leqslant N^{-1}(\rho)\left[N^{-1}\left(\frac{|u(x)|}{\rho}\right) + N^{-1}(\rho_0)\right]. \tag{20.7}$$

Let

$$\|N^{-1}(|u(x)|)\|_N \leqslant a \quad (\|u\|_\Phi \leqslant 1).$$

It then follows from (20.7) that, for $\|u\|_\Phi = \rho \geqslant \rho_0$,

$$\|N^{-1}(|u(x)|)\|_N \leqslant N^{-1}(\rho)[a + N^{-1}(\rho_0)\|\kappa(x; G)\|_N] = bN^{-1}(\rho).$$

Now (20.6) follows from this inequality inasmuch as

$$\lim_{\rho \to 0} \frac{N^{-1}(\rho)}{\rho} = 0.$$

The conditions for the existence of solutions for equation (20.2), obtained with the application of Orlicz spaces, differ from those conditions for the existence of solutions which appear when one uses the space C or L^α (see, e.g., NEMYCKIĬ [1], [2], KRASNOSEL'SKIĬ [5], and SCORZA DRAGONI [1]). As we already saw above, this is explained by the fact that completely continuous Uryson operators, which do not act in the spaces C or L^α, act in Orlicz spaces.

Thus, it follows from Theorem 19.1 that equation (20.2) has solutions for sufficiently small λ if

$$\int_G \exp |f_0(x)|^{1+\beta} dx < \infty$$

provided (see (19.21) and (19.22))

$$|k(x, y, u)| \leqslant k(x, y)e^{\alpha|u|} \quad (x, y \in G, -\infty < u < \infty), \tag{20.8}$$

and

$$|k(x, y)| \leqslant a|\ln r|^{1-\beta_0} + b. \tag{20.9}$$

In this connection, the solution $u(x)$ belongs to all spaces L_Φ^* where p.p. $\Phi(u) = e^{|u|^{1+\beta}}$ and $0 \leqslant \beta \leqslant \beta_0$.

It follows from Theorem 19.2, e.g., that equation (20.2) has solutions for all λ if $f_0(x)$ is summable and (see (19.33) and (19.34))

$$|k(x, y, u)| \leqslant k(x, y)[a + \ln(|u| + 1)] \tag{20.10}$$
$$(x, y \in G, -\infty < u < \infty)$$

and

$$\int_G \int |k(x, y)| \ln(|k(x, y)| + 1)\, dx\, dy < \infty. \tag{20.11}$$

We have shown two examples. In each of them, the operator

(20.3) cannot be considered in the spaces L^{α} without additional assumptions: in the first example — because of the "strong" nonlinearity, and in the second — because of the fact that the kernel $k(x, y)$ may have "strong" singularities.

Other fixed-point principles can also be used in the proof of theorems on the existence of solutions.

3. *Positive characteristic functions.* A closed convex set $\mathfrak{K}$ in the Banach space E is called a *cone* if $u \in \mathfrak{K}$ implies that $tu \in \mathfrak{K}$ for all $t > 0$ and if $u, -u \in \mathfrak{K}$ imply that $u = \theta$.

It is easily seen that the totality of non-negative functions in any Orlicz space forms a cone.

We write $u_1 \leqslant u_2$ if $u_2 - u_1 \in \mathfrak{K}$. The operator $\boldsymbol{A}$, acting in the space E with cone $\mathfrak{K}$, is said to be *positive* if $\boldsymbol{A}\mathfrak{K} \subset \mathfrak{K}$. The operator $\boldsymbol{A}$ is said to be *monotonic* if $u_1 \leqslant u_2$ implies that $\boldsymbol{A}u_1 \leqslant \boldsymbol{A}u_2$.

Numerous conditions for the existence of characteristic vectors for positive completely continuous operators are known (see KRASNOSEL'SKIĬ [5], Chapter V). All of them are applicable to operators acting in Orlicz spaces.

We mention one of these general theorems.

Let the positive completely continuous operator $\mathfrak{K}$ satisfy the inequality

$$\boldsymbol{A}u \leqslant \boldsymbol{K}u \quad (u \in \mathfrak{K}), \tag{20.12}$$

where $\boldsymbol{A}$ is a linear completely continuous positive operator possessing the property that for arbitrary $u \in \mathfrak{K}$ $(u \neq \theta)$ positive numbers α, β can be found such that

$$\alpha u_0 \leqslant \boldsymbol{A}u \leqslant \beta u_0. \tag{20.13}$$

Then the characteristic vectors u of the operator $\boldsymbol{K}$:

$$\boldsymbol{K}u = \lambda u, \tag{20.14}$$

corresponding to positive characteristic values λ, form in the cone $\mathfrak{K}$ a continuous branch of infinite length, i.e. on the boundary of every bounded region, containing θ, there is at least one characteristic vector of the operator $\boldsymbol{K}$ lying in the cone $\mathfrak{K}$.

Suppose the conditions of Theorem 19.1 are satisfied and let $k(x, y, u)$ satisfy the additional condition

$$k(x, y, u) \geqslant a^2 u \quad (x, y \in G,\ u \geqslant 0), \tag{20.15}$$

where $a \neq 0$. Then the operator $\boldsymbol{K}$ satisfies condition (20.12) in which

$$\boldsymbol{A}u(x) = a^2 \int_G u(x)\,dx.$$

The operator $\boldsymbol{A}$, obviously, satisfies condition (20.13), where $u_0(x) \equiv 1$.

It follows from the theorem just formulated that *under these conditions, the equation*

$$\int_G k[x, y, u(y)]\,dy = \lambda u(x) \tag{20.16}$$

has a continuum of positive solutions, corresponding, possibly, to distinct values of the positive parameter λ. These solutions form in the corresponding Orlicz space a continuous branch of infinite length. In this continuous branch, there are, in particular, solutions with arbitrarily small and with arbitrarily large norms.

Thus, for example, it is sufficient for the existence of a continuous branch of positive solutions of equation (20.16) that conditions (20.8), (20.9) and (20.15) are satisfied.

4. *Characteristic functions of potential operators.* An operator is called a *potential operator* if it is the gradient of a functional. A number of theorems on the existence of characteristic functions for a potential operator are known. The utilization of Orlicz spaces in the application of these theorems allows us to extend the class of operators that can be studied. This utilization is carried out according to the following scheme.

Let $\boldsymbol{H}$ be a linear operator acting from L^2 into the Orlicz space L^*_Φ. Suppose $\boldsymbol{F}_1(u)$ is a real functional on L^*_Φ (or on some sphere in this space). Then $\boldsymbol{F}(u) = \boldsymbol{F}_1(\boldsymbol{H}u)$ will be a functional defined on L^2 (or on some sphere in the space L^2). If the operator $\boldsymbol{H}$ is completely continuous, then it is easily seen that the functional $\boldsymbol{F}(u)$ is weakly continuous, i.e. its values converge on every weakly convergent sequence of functions in L^2.

If the functional $\boldsymbol{F}_1(u)$ is differentiable in an Orlicz space and has the operator $\boldsymbol{\Gamma}_1$ for its gradient, then the functional $\boldsymbol{F}(u)$ is differentiable in L^2 and its gradient is the operator

$$\boldsymbol{\Gamma} = \boldsymbol{H}^*\boldsymbol{\Gamma}_1\boldsymbol{H}, \tag{20.17}$$

where $\boldsymbol{H}^*$ is the operator adjoint to $\boldsymbol{H}$ which acts from the conjugate space $L_{\boldsymbol{\Phi}}^*$ into L^2. In fact, if

$$\boldsymbol{F}_1(v+g) - \boldsymbol{F}_1(v) - (\boldsymbol{\Gamma}_1 v, g) = \omega(v, g),$$

where

$$\lim_{\|g\|_{\boldsymbol{\Phi}} \to 0} \frac{|\omega(v, g)|}{\|g\|_{\boldsymbol{\Phi}}} = 0,$$

then

$$\boldsymbol{F}(u+h) - \boldsymbol{F}(u) - (\boldsymbol{\Gamma}u, h) =$$

$$= \boldsymbol{F}_1(\boldsymbol{H}u + \boldsymbol{H}h) - \boldsymbol{F}_1(\boldsymbol{H}u) - (\boldsymbol{\Gamma}_1 \boldsymbol{H}u, \boldsymbol{H}h) = \omega(\boldsymbol{H}u, \boldsymbol{H}h),$$

where

$$\lim_{\|h\|_{L^2} \to 0} \frac{|\omega(\boldsymbol{H}u, \boldsymbol{H}h)|}{\|h\|_{L^2}} \leqslant \|\boldsymbol{H}\| \lim_{\|\boldsymbol{H}h\|_{\boldsymbol{\Phi}} \to 0} \frac{|\omega(\boldsymbol{H}u, \boldsymbol{H}h)|}{\|\boldsymbol{H}h\|_{\boldsymbol{\Phi}}} = 0.$$

If $u(x)$ is a solution of the equation

$$\boldsymbol{H}^* \boldsymbol{\Gamma}_1 \boldsymbol{H} u(x) = \lambda u(x), \tag{20.18}$$

then $\boldsymbol{H}u(x)$ will be a solution of the equation

$$\boldsymbol{H}\boldsymbol{H}^* \boldsymbol{\Gamma}_1 v(x) = \lambda v(x). \tag{20.19}$$

Thus, the proof of theorems on the existence of distinct solutions of equation (20.19) can be reduced to the proof of theorems on the existence of solutions in L^2 of equation (20.18). But, in the proof of the existence of solutions in L^2 of equations with potential operator one can apply, as we have already pointed out, various general propositions (see KRASNOSEL'SKIĬ [5], Chapter 6).

Theorems 16.8 and 18.1 contain the conditions under which the operator

$$\boldsymbol{K}u(x) = \int_G k(x, y) f[y, u(y)]\, dy \tag{20.20}$$

with symmetric positive definite kernel $k(x, y)$ is representable in the form $\boldsymbol{K} = \boldsymbol{H}\boldsymbol{H}^* \boldsymbol{\Gamma}_1$, where $\boldsymbol{\Gamma}_1 = \boldsymbol{f}$ is the gradient of a functional.

It is known that the gradient of every weakly continuous functional in L^2 has a continuum of characteristic functions. Therefore, it follows from the above discussion that *if the conditions*

of Theorems 16.8 *and* 18.1 *are satisfied, then the equation*

$$\int_G k(x, y)f[y, u(y)]\,dy = \lambda u(y)$$

has a continuum of distinct characteristic functions, each of which corresponds to a value of the parameter λ.

Variational considerations can also be applied in the proof of theorems on the existence of solutions.

5. *Theorem on branch points.* Suppose the completely continuous operator $\boldsymbol{K}$ ($\boldsymbol{K}\theta = \theta$) has the Fréchet derivative $\boldsymbol{B}$ at the point θ. In this connection, the operator $\boldsymbol{B}$ is also completely continuous. It is known that every characteristic value of odd multiplicity of the linear operator $\boldsymbol{B}$ is a branch point of the operator $\boldsymbol{K}$.

In order to apply this assertion to the investigation of nonlinear integral operators $\boldsymbol{K}$, which are completely continuous in some Orlicz space, one must know the conditions under which this operator $\boldsymbol{K}$ is differentiable at the zero of this space.

Let us consider, for example, the operator (20.20). It can be represented in the form of the composition $\boldsymbol{K} = \boldsymbol{A}\boldsymbol{f}$, where $\boldsymbol{A}$ is a linear integral operator defined by the kernel $k(x, y)$. If the operator $\boldsymbol{f}$ acts from the space $L^*_{M_1}$ into the space $L^*_{M_2}$ and has the Fréchet derivative $\boldsymbol{f}_1$ at the point θ: $\boldsymbol{f}_1 u(x) = f'_u(x, 0)u(x)$, and the operator $\boldsymbol{A}$ acts from $L^*_{M_2}$ into $L^*_{M_1}$, then, as is easily seen, the operator $\boldsymbol{K}$ is differentiable at the zero θ of the space L^*_M, and its Fréchet derivative $\boldsymbol{B}$ has the form

$$\boldsymbol{B}u(x) = \boldsymbol{A}\boldsymbol{f}_1\boldsymbol{u}(x) = \int_G k(x, y)f'_u(y, 0)u(y)\,dy. \qquad (20.21)$$

Combining the conditions for the complete continuity of a linear integral operator (see § 16) with the conditions under which the operator $\boldsymbol{f}$ acts from $L^*_{M_1}$ into $L^*_{M_2}$ and is differentiable at the zero θ of the space $L^*_{M_1}$ (see § 18), we arrive at the conditions under which every characteristic value of odd multiplicity of the kernel $k(x, y)f'_u(y, 0)$ is a branch point of the operator (20.20).

SUMMARY OF FUNDAMENTAL RESULTS

Convex functions

1. Let $p(t)$ and $q(s)$ be two right-continuous $(s, t > 0)$ non-decreasing functions which are the "inverses" of one another in the sense that

$$q(s) = \sup_{p(t) \leqslant s} t, \quad p(t) = \sup_{q(s) \leqslant t} s \tag{1}$$

and which satisfy the conditions $p(0) = q(0) = 0$,

$$p(+\infty) = q(+\infty) = +\infty.$$

The convex functions $M(u)$ and $N(v)$, defined by the equalities

$$M(u) = \int_0^{|u|} p(t)\,dt, \quad N(v) = \int_0^{|v|} q(s)\,ds, \tag{2}$$

are called mutually complementary N-functions.

The following pairs of functions:

$$M_1(u) = \frac{|u|^\alpha}{\alpha} \ (\alpha > 1), \quad N_1(v) = \frac{|v|^\beta}{\beta} \left(\frac{1}{\alpha} + \frac{1}{\beta} = 1\right),$$

$$M_2(u) = e^{|u|} - |u| - 1, \quad N_2(v) = (1 + |v|) \ln (1 + |v|) - |v|$$

can serve as examples of mutually complementary N-functions.

If $M_1(u) \leqslant M_2(u)$ for large values of the argument, then the reverse inequality $N_2(v) \leqslant N_1(v)$ is satisfied by the complementary N-functions for large values of v.

The Young inequality is valid for mutually complementary N-functions:

$$uv \leqslant M(u) + N(v). \tag{3}$$

This inequality transforms into an equality only for values of u and v which are related in a definite way:

$$|u|p(|u|) = M(u) + N[p(|u|)], \tag{4}$$

$$q(|v|)\,|v| = M[q(|v|)] + N(v). \tag{5}$$

2. If $M_1(u) \leqslant M_2(ku)$ for large values of the argument then we write $M_1(u) \prec M_2(u)$. If $M_1(u) \prec M_2(u)$ or $M_2(u) \prec M_1(u)$, then the N-functions $M_1(u)$ and $M_2(u)$ are said to be *comparable*. The relation $M_1(u) \prec M_2(u)$ implies the relation $N_2(v) \prec N_1(v)$ for the complementary N-functions.

The N-functions $M_1(u)$ and $M_2(u)$ are said to be *equivalent* (and we write $M_1(u) \sim M_2(u)$) if $M_1(u) \prec M_2(u)$ and $M_2(u) \prec \prec M_1(u)$; N-functions which are complementary to equivalent functions are also equivalent. There are various criteria for the equivalence of N-functions.

For every sequence of N-functions $M_n(u)$ $(n = 1, 2, \cdots)$ there can be found N-functions $\Phi(u)$ and $\Psi(u)$ such that the relations $\Phi(u) \prec M_n(u) \prec \Psi(u)$ are valid for all n.

3. We say that $M(u)$ *satisfies the* Δ_2*-condition* if

$$M(2u) \leqslant kM(u) \tag{6}$$

for large values of the argument. N-functions which satisfy the Δ_2-condition can be majorized by a power function for large values of the argument.

A necessary and sufficient condition that $M(u)$ satisfy the Δ_2-condition is that

$$\overline{\lim_{u\to\infty}} \frac{up(u)}{M(u)} < \infty. \tag{7}$$

A necessary and sufficient condition that $M(u)$ satisfy the Δ_2-condition is that the complementary function $N(v)$ satisfy the inequality

$$N(v) \leqslant \frac{1}{2l} N(lv), \tag{8}$$

where $l > 1$, for large values of the argument.

The functions

$$M_1(u) = \frac{|u|^\alpha}{\alpha}\ (\alpha > 1),\ M_2(u) = (1 + |u|)\ln(1 + |u|) - |u|,$$
$$M_3(u) = |u|^\alpha(|\ln|u|| + 1)\ (\alpha > 1),\ M_4(u) = \frac{u^2}{\ln(|u| + e)} \tag{9}$$

can serve as examples of N-functions which satisfy the Δ_2-condition.

The N-function $M(u)$ complementary to $N(v) = e^{v^2} - 1$ also satisfies the Δ_2-condition; the explicit form of the function $M(u)$ is unknown.

There also exist mutually complementary N-functions $M(u)$ and $N(v)$ neither of which satisfies the Δ_2-condition.

4. We say that the N-function $M(u)$ *satisfies the Δ'-condition* if

$$M(uv) \leqslant CM(u)M(v) \tag{10}$$

for large values of u and v.

If the N-function $M(u)$ satisfies the Δ'-condition, then it also satisfies the Δ_2-condition.

For example, the N-functions $M_1(u)$, $M_2(u)$ and $M_3(u)$ in (9) satisfy the Δ'-condition. The function $M_4(u)$ does not satisfy the Δ'-condition.

If the function

$$h(t) = \frac{p(ut)}{p(t)} \tag{11}$$

does not increase for large t for every fixed, sufficiently large, u, then the N-function $M(u) = \int_0^{|u|} p(t)\,dt$ satisfies the Δ'-condition.

If the function $p(t)$ is differentiable for large values of t, then a sufficient condition for the Δ'-condition to be fulfilled is that the function

$$g(t) = \frac{tp'(t)}{p(t)} \tag{12}$$

does not increase for large values of t. If $g(t)$ does not decrease, then the N-function $N(v)$ complementary to $M(u)$ satisfies the Δ'-condition.

5. If the N-function $M(u)$ is equivalent to $|u|M(u)$, then we say that $M(u)$ *satisfies the* Δ_3*-condition.*

Functions which satisfy the Δ_3-condition increase more rapidly than any power function for large values of the argument. However, not all N-functions which increase more rapidly than an arbitrary power function satisfy the Δ_3-condition.

If $M(u)$ satisfies the Δ_3-condition, then the N-function $N(v)$ complementary to it satisfies the Δ_2-condition and it satisfies the inequalities

$$k_1 v M^{-1}(k_1 v) \leqslant N(v) \leqslant k_2 v M^{-1}(k_2 v) \tag{13}$$

for large values of the argument, where $M^{-1}(v)$ is the function inverse to $M(u)$.

If $M(u)$ satisfies the Δ_3-condition and

$$2p^2(u) \geqslant M(u)p'(u) \tag{14}$$

for large values of the argument, then the N-function $N(v)$ complementary to $M(u)$ is equivalent to the N-function which equals $vM^{-1}(v)$ for large values of the argument. For example, if $M(u) = e^{u^2} - 1$, then $N(v)$ is equivalent to the N-function which equals $v\sqrt{\ln v}$ for large values of v.

If $M(u)$ satisfies the Δ_3-condition, then $M(u) \sim N[M(u)]$.

From the class of N-functions satisfying the Δ_3-condition, we select a narrower class of functions, *satisfying the* Δ^2*-condition*: $M(u) \sim M^2(u)$. The N-functions $M_1(u) = e^{|u|} - |u| - 1$, $M_2(u) = e^{u^2} - 1$ can serve as examples of such functions. The N-function $M(u)$ which equals $u^{\ln u}$ for large values of u satisfies the Δ_3-condition, but it does not satisfy the Δ^2-condition.

A sufficient condition for $M(u)$ to satisfy the Δ^2-condition is that the inequality $p^2(u) < p(ku)$ be satisfied for large values of the argument. A necessary and sufficient condition that the N-function $M(u)$ satisfy the Δ^2-condition is that the N-function complementary to $M(u)$ satisfy the inequality

$$\frac{N(v)}{v} < k\frac{N(\sqrt{v})}{\sqrt{v}} \tag{15}$$

for large values of v.

If $M(u)$ satisfies the Δ^2-condition, then the complementary function $N(v)$ satisfies the Δ'-condition.

If the N-functions $M_1(u)$ and $M_2(u)$ increase more rapidly than an arbitrary power function, then, under certain additional assumptions, the composition $N_1[N_2(v)]$ is equivalent to the N-function $N_1(v)N_2(v)/|v|$. This occurs, for example, if both the functions $M_1(u)$ and $M_2(u)$ satisfy the Δ^2-condition.

6. Suppose the N-function $M(u)$ coincides with a function of the form

$$u^{\alpha}(\ln u)^{\gamma_1}(\ln \ln u)^{\gamma_2} \cdots (\ln \ln \cdots \ln u)^{\gamma_n} \quad (\alpha > 1)$$

for large values of the argument. Then the function $N(v)$ which is complementary to $M(u)$ is equivalent to the N-function

$$v^{\beta}[(\ln v)^{\gamma_1}(\ln \ln v)^{\gamma_2} \cdots (\ln \ln \cdots \ln v)^{\gamma_n}]^{1-\beta},$$

where $1/\alpha + 1/\beta = 1$, for large values of the argument.

Orlicz spaces

1. Suppose G is a bounded closed set in a finite-dimensional Euclidean space and that $M(u)$ is an N-function. The totality of functions $u(x)$ for which

$$\rho(u; M) = \int_G M[u(x)]\,dx < \infty \tag{16}$$

is called the *Orlicz class* $L_M = L_M(G)$.

Every function, which is summable on G, belongs to some Orlicz class.

Orlicz classes are convex sets. The class L_M is a linear set if, and only if, the N-function $M(u)$ satisfies the Δ_2-condition.

A sequence of functions $u_n(x) \in L_M$ is said to be mean convergent to zero if $\lim_{n\to\infty} \rho(u_n; M) = 0$.

2. The linear hull of the Orlicz class L_M transforms into a complete normed space L_M^* if a norm is introduced by means of the equality

$$\|u\|_M = \sup_{\rho(v;\, M)\leqslant 1} \int_G u(x)v(x)\,dx. \tag{17}$$

The space L_M^* is called an *Orlicz space*. This space is separable if, and only if, the N-function $M(u)$ satisfies the Δ_2-condition.

The norm of the characteristic function $\kappa(x; \mathscr{E})$ of the set $\mathscr{E} \subset G$ is calculated by means of the formula

$$\|\kappa(x; \mathscr{E})\|_M = \text{mes } \mathscr{E} N^{-1}\left(\frac{1}{\text{mes } \mathscr{E}}\right), \tag{18}$$

where $N^{-1}(u)$ is the function inverse to the N-function $N(v)$ which is complementary to $M(u)$.

To calculate the norm, one can make use of the formula

$$\|u\|_M = \inf_{k>0} \frac{1}{k}\left(1 + \int_G M[ku(x)]\,dx\right). \tag{19}$$

A norm equivalent to the norm (17) can be introduced into the space L_M^*; this is the so-called *Luxemburg norm*:

$$\|u\|_{(M)} = \inf k,$$

where the infimum ranges over those positive k for which

$$\rho\left(\frac{u}{k}; M\right) = \int_G M\left[\frac{u(x)}{k}\right] dx \leqslant 1.$$

The Luxemburg norm is connected with the Orlicz norm by means of the inequalities

$$\|u\|_{(M)} \leqslant \|u\|_M \leqslant 2\|u\|_{(M)}.$$

These norms differ by a constant factor only in the case when L_M^* is the space L^α of functions which are α-summable for some power $\alpha > 1$.

3. The inequalities

$$\|u\|_M \leqslant 1 + \int_G M[u(x)]\,dx \tag{20}$$

and

$$\int_G M\left[\frac{u(x)}{\|u\|_M}\right] dx \leqslant 1 \tag{21}$$

hold.

If $\|u\|_M \leqslant 1$, then

$$\int_G M[u(x)]\,dx \leqslant \|u\|_M. \tag{22}$$

For any pair of functions $u(x) \in L_M^*$, $v(x) \in L_N^*$, the function $u(x)v(x)$ is summable and the *Hölder inequality*

$$\int_G u(x)v(x)\,dx \leqslant \|u\|_M \|v\|_N \tag{23}$$

is valid.

A sufficient condition for the product $u(x)w(x)$ of the functions $u(x) \in L_M^*$, $w(x) \in L_\Phi^*$ to belong to the space $L_{M_2}^*$ is that there exist mutually complementary functions $R(u)$ and $Q(u)$ such that the inequalities

$$\begin{aligned} R(\alpha u) &< M_2^{-1}[M_1(u)], \\ Q(\alpha u) &< M_2^{-1}[\Phi(u)] \end{aligned} \tag{24}$$

are satisfied for large values of the argument. If the N-function $M_2(u)$ satisfies the Δ'-condition, then it is sufficient that the inequalities

$$\begin{aligned} R(\alpha u) &< M_1[M_2^{-1}(u)], \\ Q(\alpha u) &< \Phi[M_2^{-1}(u)] \end{aligned} \tag{25}$$

hold.

The inequality

$$\|u(x)w(x)\|_{M_2} \leqslant k\|u\|_{M_1}\|w\|_\Phi$$

holds in the cases described above.

The space $L_M^*(\hat{G})$, where $\hat{G}$ is the topological product $G \times G$, is denoted by $\hat{L}_M^*$. A necessary and sufficient condition that, for an arbitrary pair of functions $u(x)$, $v(x) \in L_M^*$, the product $u(x)v(x)$ belong to $\hat{L}_M^*$ is that the N-function $M(u)$ satisfy the Δ'-condition. In this connection,

$$\|u(x)v(y)\|_{\hat{M}} \leqslant c\|u\|_M \|v\|_M.$$

4. The convergence of the sequence $u_n(x)$ to the function $u_0(x)$ with respect to the norm in the space L_M^* implies the mean convergence to zero of the sequence $u_n(x) - u_0(x)$. Convergence in

norm is equivalent to mean convergence to zero of the sequence $u_n(x) - u_0(x)$ if, and only if, $M(u)$ satisfies the Δ_2-condition.

5. If $M(u)$ does not satisfy the Δ_2-condition, then the set of bounded functions is nowhere dense in the space L_M^*. The closure in L_M^* of the set of bounded functions is denoted by E_M. This space plays an important role. It coincides with $L_M^* = L_M$ if $M(u)$ satisfies the Δ_2-condition.

E_M is separable and has a basis.

A necessary and sufficient condition that the function $u(x) \in L_M^*$ belong to E_M is that its norm be absolutely continuous. The absolute continuity of the norm signifies that to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $\|u(x)\kappa(x; \mathscr{E})\|_M < \varepsilon$ provided mes $\mathscr{E} < \delta$ ($\mathscr{E} \subset G$).

If $p(u) = M'(u)$ is continuous then to calculate the norm of a function in E_M one can use the formula

$$\|u\|_M = \int_G p(k^* |u(x)|)\, |u(x)|\, dx, \tag{26}$$

where k^* is determined from the equation

$$\int_G N[p(k^* |u(x)|)]\, dx = 1. \tag{27}$$

The space E_M enables us to describe the disposition of the class L_M in the space L_M^*: the class L_M contains the totality Π of all functions u for which $\inf_{w \in E_M} \|u - w\|_M < 1$ and is contained in the closure $\overline{\Pi}$. If E_M is a proper subset of the space L_M^* (i.e. $M(u)$ does not satisfy the Δ_2-condition), then Π is a proper subset of L_M and L_M is a proper subset of $\overline{\Pi}$.

6. Under natural assumptions, the Orlicz norm and the Luxemburg norm are differentiable in the space E_M (or in the space $L_M^* = L_M$ if $M(u)$ satisfies the Δ_2-condition).

Suppose the N-function $N(v)$, which is complementary to $M(u)$, satisfies the Δ_2-condition. Suppose the N-function $M(u)$ has a continuous monotonic second derivative which is positive for $u \neq 0$. Then the Orlicz norm is a differentiable functional on E_M. The gradient Γ of the Orlicz norm is defined by the equality

$$\Gamma u(x) = p(k^* u(x)), \tag{28}$$

where

$$\int_G N[p(k^*|u(x)|)]\,dx = 1.$$

The gradient Γ_1 of the Luxemburg norm $\|u\|_{(M)}$ is defined by the equality

$$\Gamma_1 u(x) = \frac{p\left(\frac{u(x)}{\|u\|_{(M)}}\right)}{\int_G p\left(\frac{u(x)}{\|u\|_{(M)}}\right)\frac{u(x)}{\|u\|_{(M)}}\,dx}. \tag{29}$$

7. We say that a family $\mathfrak{M} \subset L_M^*$ has *equi-absolutely continuous norms* if for every $\varepsilon > 0$ a $\delta > 0$ can be found such that for all functions $u(x)$ of the family we have that

$$\|u(x)\kappa(x; \mathscr{E})\|_M < \varepsilon \text{ provided mes } \mathscr{E} < \delta.$$

A necessary and sufficient condition that a sequence of functions $u_n(x) \in E_M$ which converges in measure also be convergent in norm is that it have equi-absolutely continuous norms. It follows from this fact that the family $\mathfrak{M} \subset E_M$ is compact in L_M^* if it has equi-absolutely continuous norms and is compact in the sense of convergence in measure.

The known criteria for compactness due to A. N. Kolmogorov and F. Riesz also generalize to families of functions situated in the space E_M.

8. Generally speaking, distinct N-functions determine distinct Orlicz spaces.

A necessary and sufficient condition that the set-theoretic inclusion $L_{M_1}^* \subset L_{M_2}^*$ hold is that the relation $M_2(u) \prec M_1(u)$ be satisfied. In this connection, the norms turn out to be comparable:

$$\|u\|_{M_2} \leqslant q\|u\|_{M_1} \quad (u(x) \in L_{M_1}^*). \tag{30}$$

The spaces $L_{M_1}^*$ and $L_{M_2}^*$ consist of the same functions if, and only if, $M_1(u) \sim M_2(u)$. Norms generated by equivalent N-functions are equivalent.

Functionals and operators

1. The general form of a functional on the space E_M is given by the formula

$$l(u) = \int_G u(x)v(x)\,dx, \tag{31}$$

where $v(x) \in L_N^*$. The norm of the functional l coincides with the Luxemburg norm $\|v\|_{(N)}$ of the function $v(x)$.

If $M(u)$ does not satisfy the Δ_2-condition, then there exist linear functionals on L_M^* which do not admit of an integral representation (31).

The space L_M^* is reflexive if, and only if, $M(u)$ and the complementary function $N(v)$ satisfy the Δ_2-condition.

2. E_N-weak convergence is introduced in the space L_M^* as follows: the sequence of functions $u_n(x) \in L_M^*$ is *E_N-weakly convergent* if the sequence of numbers $\int_G u_n(x)v(x)\,dx$ converges for every function $v(x) \in E_N$. The E_N-weak convergence of a sequence of functions implies the boundedness of the norms of the elements of the sequence.

Every Orlicz space is E_N-weakly complete and E_N-weakly compact. The space E_M does not possess these properties: the E_N-weak closure of the space E_M is the entire space L_M^*.

Convergence in measure and boundedness of the norm imply E_N-weak convergence.

3. Various criteria have been found for the continuity and complete continuity of linear integral operators

$$A\varphi(x) = \int_G k(x, y)\varphi(y)\,dy \tag{32}$$

acting from one Orlicz space into another. We formulate the fundamental result.

Let $\Phi(u)$ and $\Psi(v)$ be mutually complementary N-functions. Suppose the kernel $k(x, y)$ belongs to the space L_Ψ^*. Then the operator (32) acts from the space $L_{M_1}^*$ into the space $L_{M_2}^*$ and is

continuous if any one of the following three conditions is satisfied:

a) $M_2[N_1(v)] \prec \Psi(v)$;
b) $N_1[M_2(v)] \prec \Psi(v)$;
c) the function $\Phi(u)$ satisfies the Δ'-condition and

$$M_2(v) \prec \Psi(v), \quad N_1(v) \prec \Psi(v).$$

If, under the conditions of this theorem, the kernel $k(x, y)$ belongs to the space $\hat{E}_\Psi$ (i.e. to the closure in $\hat{L}^*_\Psi$ of the set of functions which are bounded on G), then the operator (28) is completely continuous. In this case, the operator (32) transforms every E_N-weakly convergent sequence of functions in $L^*_{M_1}$ into a sequence which converges with respect to the norm of the space $L^*_{M_2}$.

4. Let $M(u)$ and $N(v)$ be mutually complementary N-functions where $N(u) \prec u^2 \prec M(u)$. Suppose the square $\boldsymbol{A}^2$ of a positive definite linear continuous operator $\boldsymbol{A}$, which is self-adjoint in L^2, is continuously extendible to an operator which acts from E_N into L^*_M. Then the operator $\boldsymbol{A}$ acts from L^2 into L^*_M and is continuous.

If, under the conditions of the theorem formulated above, the extension of the operator $\boldsymbol{A}^2$ is a completely continuous operator, acting from E_N into L^*_M, then the operator $\boldsymbol{A}$ is also a completely continuous operator, acting from L^2 into L^*_M.

These assertions imply various conditions for the decomposition of a linear operator $\boldsymbol{A}$, acting from E_N into L^*_M, into the product $\boldsymbol{A} = \boldsymbol{H}\boldsymbol{H}^*$, where $\boldsymbol{H}$ acts from L^2 into L^*_M and $\boldsymbol{H}^*$ is the operator adjoint to $\boldsymbol{H}$ and acting from E_N into L^2.

5. We investigate the operator $\boldsymbol{f}$: $\boldsymbol{f}u(x) = f[x, u(x)]$, where $f(x, u)$ $(x \in G, -\infty < u < \infty)$ is continuous in u and measurable in x for every u. Conditions are found under which the operator $\boldsymbol{f}$, acting from some region of the space $L^*_{M_1}$ into the space $L^*_{M_2}$, is continuous and bounded. In distinction from the case of L^α spaces, the operator $\boldsymbol{f}$ can be defined on a sphere but not be defined in the entire space. It can be continuous at every point of some bounded closed region but not be bounded in this region itself.

We shall formulate several propositions on the properties of the operator $\boldsymbol{f}$.

We denote the totality of functions $u(x) \in L^*_{M_1}$ for which the distance to E_{M_2} is less than r by Π_r. Suppose the operator $\boldsymbol{f}$ acts from Π_r into E_{M_2}. Then the operator $\boldsymbol{f}$ is continuous at every point

of Π_r. The set of its values on the sphere $\|u\|_{M_1} \leqslant r_1 < r$ is bounded with respect to the norm in $L^*_{M_2}$.

It is convenient to express concrete conditions for the continuity and boundedness of the operator $\boldsymbol{f}$ in terms of estimates of the growth of the function $f(x, u)$. If

$$|f(x, u)| \leqslant b(x) + aQ^{-1}\left\{M_2^{-1}\left[M_1\left(\frac{u}{r}\right)\right]\right\} \tag{33}$$

$$(x \in G, \ -\infty < u < \infty),$$

where $b(x) \in E_{M_2}$, $Q(u)$ is an N-function and a, $r > 0$. then the operator $\boldsymbol{f}$ which acts from Π_r into E_{M_2} is continuous at all points of Π_r and is bounded on every sphere $\|u\|_{M_1} \leqslant r_1 < r$. In the case when $M_2(u)$ satisfies the Δ_2-condition, we can set $Q(u) = u$ in (33).

6. Suppose there exist mutually complementary N-functions $R(u)$ and $Q(u)$ such that for large values of the argument inequalities (24) are satisfied or, if $M_2(u)$ satisfies the Δ'-condition, the inequalities (25) are satisfied. Suppose there exists a derivative $f'_u(x, u)$ which defines a continuous operator $\boldsymbol{f}_1 : \boldsymbol{f}_1 u(x) = f'_u[x, u(x)]$, which acts from the sphere $\|u\|_{M_1} \leqslant r$ into the space L^*_{Φ}. Under these assumptions, the operator $\boldsymbol{f}$ is differentiable in the Fréchet sense at every interior point of the indicated sphere, where its Fréchet differential $\boldsymbol{B}h$ at the point $u(x)$ has the form

$$\boldsymbol{B}h(x) = f'_u[x, u(x)]h(x).$$

The conditions for the differentiability of the operator $\boldsymbol{f}$ not in a region but at an isolated point are more delicate.

7. The nonlinear integral operator

$$\boldsymbol{K}\varphi(x) = \int_G k(x, y)f[y, \varphi(y)]\,dy \tag{34}$$

can be represented as the composition of a nonlinear operator $\boldsymbol{f}$ and a linear integral operator (32). Combining the conditions under which the operator $\boldsymbol{f}$ acts from the space $L^*_{M_1}$ into the space $L^*_{M_2}$ and is continuous and bounded on a sphere with the conditions under which operator (32) acts from $L^*_{M_2}$ into $L^*_{M_1}$ and is completely continuous, we arrive at the conditions for complete continuity of the operator (34) in the space $L^*_{M_1}$.

Conditions for complete continuity in Orlicz spaces are also established for nonlinear integral operators of a more general form:

$$\boldsymbol{K}\varphi(x) = \int_G k[x, y, \varphi(y)]\,dy. \tag{35}$$

These conditions enable us, for certain operators with essentially non-power nonlinearities, to choose Orlicz spaces in which they are completely continuous.

Suppose, for example, that

$$|k(x, y, u)| \leqslant k(x, y)[a(x) + R(|u|)$$

$$(x, y \in G, -\infty < u < \infty),$$

where $k(x, y) \in \hat{E}_M$, $a(x) \in L_N^*$, $R(u)$ is a non-negative continuous function, $M(u)$ satisfies the Δ^2-condition (e.g., it increases as e^u). Suppose that $R(\gamma u) \leqslant KM(u)$ for large values of the argument.

Under these assumptions, the operator (35) acts from some sphere in the space L_M^* into L_M^* and is completely continuous.

A number of other properties of nonlinear integral operators are proved.

8. Knowledge of the spaces in which integral operators (35) possess "good" properties (i.e. they are continuous, completely continuous, differentiable, potentials, and so on) enables us to apply the general methods of nonlinear functional analysis to the investigation of the equation

$$\lambda\varphi(x) = \int_G k[x, y, \varphi(y)]\,dy. \tag{36}$$

The application of these methods leads to various theorems on the existence of solutions and characteristic functions, to theorems on branch points and structure of the spectrum, and so on. A peculiarity of the theorems proved consists in this that the nonlinearities in the equations considered can be of an essentially non-power character.

BIBLIOGRAPHICAL NOTES

§§ 1, 2. The fundamental concepts of the theory of convex functions were established by Jensen (see JENSEN [1]). The reader can find detailed discussions of the elements of the theory of convex functions (in particular of N-functions) in HARDY, LITTLEWOOD and PÓLYA [1], PÓLYA and SZEGÖ [1], and ZYGMUND [1].

The definition of an N-function which we have accepted coincides with the definition of an N'-function in BIRNBAUM and ORLICZ [1].

§ 3. BIRNBAUM and ORLICZ [1] call the N-functions $M_1(u)$ and $M_2(u)$ *equivalent* if $aM_1(u) \leqslant M_2(u) \leqslant bM_1(u)$ for large values of the argument. Equivalence in this sense signifies that the N-functions $M_1(u)$ and $M_2(u)$ determine the same Orlicz class (see § 8). Equivalence in our definition (see KRASNOSEL'SKIĬ and RUTICKIĬ [1]) means that the N-functions $M_1(u)$ and $M_2(u)$ determine Orlicz spaces consisting of the same functions.

§§ 4, 5. It appears that classes of N-functions satisfying the Δ_2-condition and Δ'-condition were first isolated in the paper by BIRNBAUM and ORLICZ [1]. Criteria for the fulfillment of these conditions did not interest them.

The propositions pointed out in §§ 4, 5 were published earlier in KRASNOSEL'SKIĬ and RUTICKIĬ [6], [7]. An assertion, close to Theorem 4.2, was formulated by S. M. LOZINSKIĬ [1] (under the additional assumption that $N(v)$ satisfies the Δ_2-condition). (When S. M. Lozinskiĭ became acquainted with the manuscript of the present book, he graciously notified the authors of a number of tests found by him for the fulfillment of the Δ_2-condition; these investigations, executed about ten years ago, have still not been published.)

We note that up to the present time, sufficiently useful necessary and sufficient criteria for the fulfillment of the Δ'-condition have not been found. It would be interesting to find such criteria in terms of the complementary function.

§ 6. In this section, with essential additions, are discussed propositions which were published earlier (see KRASNOSEL'SKIĬ and RUTICKIĬ [3], [6], [9]).

We know suitable criteria that the N-function $M(u)$ satisfy the Δ_3-condition, expressed in terms of the complementary function $N(v)$. It would be desirable to obtain such criteria in a form analogous to Theorem 6.8.

Narrower classes of rapidly increasing N-functions can be isolated with the aid of a condition analogous to the Δ^2-condition. We shall say, for example, that the N-function $M(u)$ satisfies the Δ_Φ-condition, where $\Phi(u)$ is a fixed N-function, if $\Phi[M(u)] \sim M(u)$. It can be shown that the class of N-functions, satisfying every Δ_Φ-condition, is non-empty. For example, if p. p. $\Phi(u) = e^{\ln^2 u}$, p. p. $M(u) = e^{e^u}$, then, obviously, $\Phi[M(u)] \sim M(u)$. A study of classes of N-functions which satisfy the Δ_Φ-condition has not been made.

§ 7. The results of this section were discussed earlier in KRASNOSEL'SKIĬ and RUTICKIĬ [4] for a somewhat more general class of N-functions.

§ 8. The classes of functions under consideration were studied in detail in the paper by BIRNBAUM and ORLICZ [1] (also see YOUNG [1]). Orlicz classes and spaces were used in a number of works by various authors (see, for example, ZYGMUND [1], KALUGINA [1–3], KIPRIYANOV [1–2], KORENBLYUM [1], LOZINSKIĬ [1], MEDVEDEV [1]) basically in connection with problems of the theory of functions (theory of series, singular integrals, approximation theory, and so on). Orlicz classes were utilized in connection with the theory of partially ordered spaces by KANTOROVIČ, VULIH and PINSKER [1].

We also note the connection of the works of the Japanese mathematicians NAKANO [1], YAMAMURO [1], AMEMIYA [1], and others, on the theory of modulared spaces, with the theory of Orlicz classes and spaces.

§ 9. The L_M^* spaces were considered by ORLICZ [1]. In this article, L_M^* spaces were studied under the assumption that the N-function $M(u)$ satisfies the Δ_2-condition. A number of properties of L_M^* spaces are shown, without this assumption, in ORLICZ [2] and in ZYGMUND [1].

The norm, defined by formula (9.18), was introduced by LUXEM-

BURG [1]. Norms can be introduced in modulared spaces by an analogous formula (see NAKANO [1] and YAMAMURO [1]).

We note that a number of authors studied the Orlicz space of functions defined on sets of infinite measure and on sets whose measure is not continuous (see, for example, ZYGMUND [1], LUXEMBURG [1], SOBOLEV [2]). In the particular case when the set consists of sequences of points each of which has a measure equal to unity the Orlicz space is transformed into the space of numerical sequences. This space is denoted by l^M; it is a generalization of the spaces l^p. These spaces were studied by ORLICZ [2] and recently by the young Kazan mathematician Yu. I. Gribanov. DINCULEANU [1–2] considered Orlicz spaces of abstract functions.

§ 10. The space E_M was introduced and studied by KRASNOSEL'SKIĬ and RUTICKIĬ [2, 5, 6]. Necessary and sufficient conditions for the separability of Orlicz spaces were shown in ORLICZ [2]. Theorems 10.1 and 10.3 were proved in KRASNOSEL'SKIĬ and RUTICKIĬ [2, 6]. Subsection 7 was written jointly by the authors and N. G. Šimko.

§ 11. The generalization of A. M. Kolmogorov's theorem on the conditions for compactness of families of functions in L^p spaces to the case of Orlicz spaces was carried out by TAKAHASHI [1] under the assumption that the N-function $M(u)$ satisfies the Δ_2-condition. More accurately: Theorem 11.2 is due to him. Theorem 11.3 for the case of L^p spaces is shown in KRASNOSEL'SKIĬ [3]. Theorem 11.4, which is a generalization of the analogous theorem of F. Riesz for L^p spaces (see RIESZ [1]), was proved by GRIBANOV [1]. We note that an assertion close to Theorem 11.4 can be obtained as the immediate consequence of a general compactness criterion proved by ŠILOV [2].

We consider the linear normed space R consisting of complex-valued functions defined on a commutative bicompact group H (with addition as the operation). According to ŠILOV [1], the space R is called a *homogeneous function space* if, together with each function $f(t) \in R$, all its translates $f(t + h)$ also belong to the space R, where

$$\|f(t + h)\| = \|f(t)\| \quad (f(t) \in R,\ h \in H)$$

and

$$\lim_{h \to 0} \|f(t + h) - f(t)\| = 0 \quad (f(t) \in R).$$

ŠILOV'S THEOREM. *The set $\mathfrak{M}$ is compact in the homogeneous function space R if, and only if, $\mathfrak{M}$ is bounded in R and the elements of the set $\mathfrak{M}$ are equi-continuous with respect to translation.*

By *equi-continuity* we understand the following property: for arbitrary $\varepsilon > 0$ there exists a neighborhood U of the zero of the group H such that $h \in U$ implies that $\|f(t+h) - f(t)\| < \varepsilon$ for arbitrary function $f(t) \in \mathfrak{M}$.

The proof of Šilov's theorem utilizes harmonic analysis on groups.

I. P. Natanson noted that the proof in TAMARKIN [1] of the independence of the first condition in A. N. Kolmogorov's theorem on the compactness of families of functions in L^p from the remaining conditions of this theorem was erroneous. As V. N. Sudakov recently showed, the condition of the boundedness of the norm of the set of functions considered in A. N. Kolmogorov's theorem and in its subsequent generalizations to L spaces (cf. TULAĬKOV [1]) and Orlicz spaces is a consequence of the other conditions and can be omitted.

§ 12. The fact that the Haar functions form a basis in the space L_M^* if $M(u)$ satisfies the Δ_2-condition was shown by ORLICZ [1].

It is shown in the paper by SOLOMYAK [1] that a basis of Haar functions in separable Orlicz spaces possesses the orthogonality property. This signifies that the inequality

$$\|\sum_{i=1}^{n+m} c_i\varphi_i(x)\|_M \geqslant \|\sum_{i=1}^{n} c_i\varphi_i(x)\|_M$$

is satisfied for the partial sums of the series (12.3).

The second proof of the necessity condition for the separability of L_M^*, presented in § 12, is adapted from KRASNOSEL'SKIĬ and SOBOLEV [1]. The passage to the space of functions defined on a closed interval, utilized in this proof, can also be used in the proof of many other propositions. Thus, in a number of sections, we could have limited ourselves to the consideration of Orlicz spaces of functions defined on a closed interval. The authors did not do this for the reason that the consideration of arbitrary sets G with continuous measure does not give rise to any additional difficulties.

For a proof of the fact that an arbitrary set of finite continuous measure can be mapped in a one-to-one fashion onto a closed

interval in such a way that under this mapping the measure of every subset remains invariant see ROHLIN [1].

§ 13. Some of the results of this section were discussed earlier in KRASNOSEL'SKIĬ and RUTICKIĬ [1, 4, 6].

§ 14. The theorem on the general form of a linear functional on the space L_M^* in the case when the N-function $M(u)$ satisfies the Δ_2-condition was proved in ORLICZ [1]. Theorem 4.1 is proved in ORLICZ [2]. For the case when the condition is not satisfied, the theorem on the general form of a linear functional on E_M was proved in KRASNOSEL'SKIĬ and RUTICKIĬ [5] (also see LUXEMBURG [1]). The problem of the general form of a linear functional on L_M^* for the case when $M(u)$ does not satisfy the Δ_2-condition remains open.

The theorem on the connection between the norm of a linear functional and the Luxemburg norm, generated by the Luxemburg function, is proved in LUXEMBURG [1]. The function $k(v)$ is studied in SALEHOV [1, 2].

In the paper by AMEMIYA [1], in connection with the theory of modulared spaces, the relation between the Luxemburg norm and the norm defined by formula (10.11) is studied. It is assumed that these norms differ by a constant factor and it is proved that in this case the spaces considered are L^p. In virtue of Theorem 10.5, formula (10.11) defines the ordinary Orlicz norm, and the Luxemburg norm coincides with the norm of the linear functional generated by it (see subsection 5). Therefore, D. V. Salehov's theorem, introduced on p. 126, also follows from Amemiya's results.

The consideration of ordinary weak convergence is inconvenient since the general form of a linear functional on L_M^* is unknown. In this connection, it turned out to be convenient to consider that weak convergence which arises if L_M^* is assumed to be the space of linear functionals on E_N.

§§ 15, 16. We shall point out some of the numerous papers in which linear integral operators are studied. The most detailed analysis of linear integral operators for the case of space C of continuous functions was carried out by RADON [1]. The simplest theorems in the case of the spaces L^p are given in BANACH [1] and RIESZ-SZ.-NAGY [1].

Linear integral operators of a special form (the so-called potential-

type operators) are studied in the works by S. V. SOBOLEV and V. I. KONDRAŠOV (see S. SOBOLEV [1]). Strong results in the study of integral operators, acting in L^p, were obtained by KANTOROVIČ [1] who generalized the results of S. L. Sobolev and V. I. Kondrašov.

A number of theorems on linear integral operators, acting on Orlicz spaces, are proved in ZAANEN [1]. The papers by KRASNOSEL'SKIĬ and RUTICKIĬ [2, 7], the expansion of which constitutes the basic content of §§ 15 and 16, are devoted to the investigation of linear integral operators acting in Orlicz spaces.

Theorem 16.3 was proved earlier by Zaanen under the assumption that all the N-functions under consideration satisfy the Δ_2-condition.

It is natural to expect that new conditions for the continuity and complete continuity of linear integral operators acting from one Orlicz space into another can be obtained in the way proposed by KANTOROVIČ [1] in the study of operators acting in L^p spaces. Under these new conditions, the restrictions on the kernel must, apparently, be formulated in the following way: there exist functions $\Phi_1(u)$, $\Phi(u)$, $R_1(u)$ and $R_2(u)$ such that

$$\varphi(x) = \int_G R_1[k(x, y)]\, dy \in L^*_{\Phi_1}$$

and

$$\psi(y) = \int_G R_2[k(x, y)]\, dx \in L_{\Phi_2}.$$

Then the linear integral operator, defined by the kernel $k(x, y)$, acts from $L^*_{M_1}$ into $L^*_{M_2}$. In this connection, the functions $M_1(u)$ and $M_2(u)$ must be connected with the functions $\Phi_1(u)$, $\Phi_2(u)$, $R_1(u)$ and $R_2(u)$ by some relations (which are as yet unknown).

The first theorems on the decomposition of linear operators were pointed out in KRASNOSEL'SKIĬ [3] (also see KRASNOSEL'SKIĬ [5]). For the case of Orlicz spaces, the first theorems on the decomposition of linear operators were obtained in KRASNOSEL'SKIĬ [4] and discussed in detail in KRASNOSEL'SKIĬ and RUTICKIĬ [8]. A number of theorems on the decomposition of linear operators were obtained in VAĬNBERG [3, 4], KRASNOSEL'SKIĬ and KREĬN [1], KRASNOSEL'SKIĬ and SOBOLEV [2], SOBOLEV [3]. The method of proof of the theorem on the decomposition of linear operators, employed

in this book, was adapted from KRASNOSEL'SKIĬ and KREĬN [1]. We note that in Theorems 15.5 and 16.7 it is possible not to assume that the operator A is self-adjoint; then the operator AA^* rather than the operator A^2 must satisfy the condition of the theorems (see KRASNOSEL'SKIĬ and KREĬN [1]).

We note additions to embedding theorems follow from Theorem 16.9 on potential-type operators, according to a known scheme (see SOBOLEV [1] and KANTOROVIČ [1]). Other additions to the embedding theorems of S. L. Sobolev, V. I. Kondrašev, and S. M. Nikol'skiĭ with the utilization of the Orlicz metric are pointed out in KALUGINA [1, 2]).

Subsection 3 of section 16 is taken from RUTICKIĬ [3].

§ 17. The Carathéodory conditions were first used in the consideration of nonlinear operators in CARATHÉODORY [1].

Theorem 17.1, which generalizes the theorem of N. N. Luzin on the C-property of a measurable function, was apparently first proved in KRASNOSEL'SKIĬ and LADYŽENSKIĬ [1].

The operator f for the case of various functional spaces (in particular, for the L^p spaces) was studied by a number of authors (see, for example, NEMYCKIĬ [5], KRASNOSEL'SKIĬ [5], VAĬNBERG [2, 4]). For the case of Orlicz spaces fundamental propositions on the continuity and boundedness of the operator f were pointed out in RUTICKIĬ [1, 2], KRASNOSEL'SKIĬ and RUTICKIĬ [8]. These propositions are discussed in subsections 2–6 in a somewhat more general form.

Theorem 7.8 is taken from RUTICKIĬ [3].

§ 18. The functional (18.6) was used by a number of authors (see HAMMERSTEIN [1], GOLOMB [1], KRASNOSEL'SKIĬ [5], VAĬNBERG [1, 4], SOBOLEV [1], RUTICKIĬ [2], and others). Theorem 18.1, for the case of L^p spaces, was proved by A. I. Povolockiĭ (see KRASNOSEL'SKIĬ [2]). The functional (18.6), defined on Orlicz spaces, was considered in RUTICKIĬ [2] and later in KRASNOSEL'SKIĬ and RUTICKIĬ [8].

General facts of the differential calculus in functional spaces are discussed in LYUSTERNIK and SOBOLEV [1], HILLE [1], VAĬNBERG [1]. Some of the theorems on the differentiability of the operator f, acting in Orlicz spaces, were published earlier in KRASNOSEL'SKIĬ and RUTICKIĬ [3, 8]. Of the papers, in which the differentiability of the operator f in L^p spaces is studied, we

mention the article by VAĬNBERG [2]. Formulas for the gradients of the Orlicz norm and the Luxemburg norm (see KRASNOSEL'SKIĬ and RUTICKIĬ [10]) generalize the Mazur formula for the gradient of the norm in L^p.

§ 19. The theorems proved in this section are a generalization of some propositions obtained in KRASNOSEL'SKIĬ and LADYŽENSKIĬ [1] (also see KRASNOSEL'SKIĬ [5]). The analysis of Hammerstein operators acting in Orlicz spaces is based on the representation of these operators in the form $\boldsymbol{K} = \boldsymbol{Af}$ carried out in KRASNOSEL'SKIĬ and RUTICKIĬ [8].

The investigation of various nonlinear integral operators acting in other functional spaces was accomplished in many works (see, for instance, GOLOMB [1], SCORZA DRAGONI [1], DUBROVSKIĬ [1], VAĬNBERG [4], KRASNOSEL'SKIĬ [5], NEMYCKIĬ [1, 2], LADYŽENSKIĬ [1], and others).

§ 20. The papers by NEMYCKIĬ [2], KRASNOSEL'SKIĬ [5], and LERAY and SCHAUDER [1] are devoted to the general methods of nonlinear functional analysis.

All propositions to which references are made in this section are proved in KRASNOSEL'SKIĬ [5].

The theorem, discussed in subsection 2, on the existence of solutions for an equation with weak nonlinearities is based on reasonings due to V. M. Dubrovskiĭ, which were applied by him in DUBROVSKIĬ [1] in the study of other classes of integral equations.

The theory of Banach spaces with a cone was developed basically by M. G. Kreĭn (the fundamental propositions of this theory are discussed in KREĬN and RUTMAN [1]). The theory of cones is interrelated to a significant degree with the theory of partially ordered spaces (see KANTOROVIČ, VULIH and PINSKER [1]).

The general theorem, utilized in subsection 3, on the continuous branch of positive characteristic functions is a special case of more general theorems on operators with monotonic minorants (see KRASNOSEL'SKIĬ [5]).

The application of variational methods in the study of equations with Hammerstein operators has its beginning in the works of HAMMERSTEIN [1] and GOLOMB [1]. The scheme with the decomposition of the operator $\boldsymbol{A}$ into the product $\boldsymbol{HH}^*$, discussed in subsection 4, was proposed by one of the authors. The application

of this scheme with the use of L^p spaces is discussed in detail in KRASNOSEL'SKIĬ [5]. The indicated scheme is applied in the case of Orlicz spaces in RUTICKIĬ [2] and later in KRASNOSEL'SKIĬ and RUTICKIĬ [8].

Remark. A number of new results, related to Orlicz spaces, were obtained recently (see, e.g., ALBRYCHT [1], WEISS [1], VIDENSKIĬ [1], GRIBANOV [1], DINCULEANU [1], KRASNOSEL'SKIĬ [6], MILNES [1], and ŠRAGIN [1]).

VIDENSKIĬ [1] discovered an interesting connection between the theory of N-functions with the theory of entire functions. In particular, he showed that $N(v) \sim \ln \sum \exp[nv - M(n)]$, where $M(u)$ and $N(v)$ are arbitrary mutually complementary N-functions.

It is shown in KRASNOSEL'SKIĬ [6] that the set of functions $u(x)$ for which $M^{-1}[u(x)]$ is summable forms a ring if, and only if, $M(u)$ satisfies the Δ^2-condition.

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